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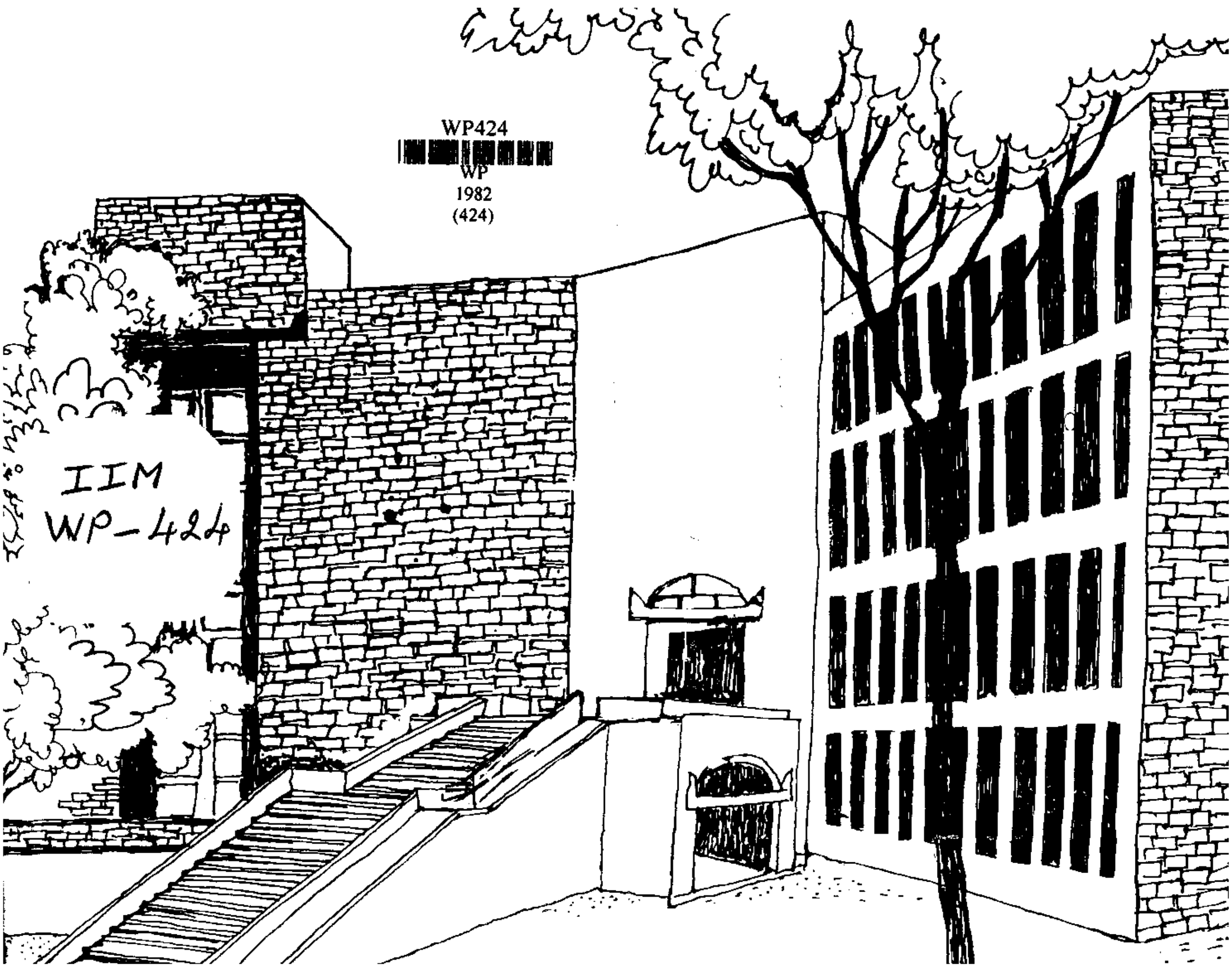

ON THE COMPUTATION OF HODGES-LEHMANN EFFICIENCY OF TEST STATISTICS

By

M. Raghavachari

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INDIAN INSTITUTE OF MANAGEMENT
AHMEDABAD-380015
INDIA

1. INTRODUCTION

Hodges and Lehmann [4] proposed an efficiency measure for the asymptotic comparison of test statistics and computed the same for a few tests in the case of normal populations. Another measure of efficiency was proposed by Bahadur [1] and since then there has been a vigorous development in this area. The study of Hodges - Lehmann efficiency has not progressed fast in comparison with Bahadur efficiency perhaps because the former is generally harder to compute. Even in simpler situations like $\bar{X}$ - test, sign test and t-tests for testing the location parameter of the normal distribution, Hodges and Lehmann [4] had to obtain precise estimates of the tail probabilities of the normal and binomial distributions when the true parameter lies in the set of alternatives. Since, in general, the distributions of test statistics under the alternatives are difficult to study analytically, the Hodges-Lehmann efficiency has not been computed for many tests. Bahadur's method of computing efficiencies is relatively easier because one has to work mostly with null distributions and in this case large deviation probabilities can be obtained with less difficulty.

Yet another approach to the computation of Bahadur efficiency was proposed in Bahadur and Raghavachari [3]. This approach enables one to compute the efficiency without obtaining large deviation probabilities explicitly. In fact it was pointed out in their paper that in some cases the large deviation probabilities can be obtained after calculating the efficiencies. This approach also applies to the calculation of Hodges and Lehmann efficiency and this was indicated in Bahadur and Raghavachari [3, P.137]. The objective of this paper is to exhibit this approach of the computation of Hodges - Lehmann efficiency for a few examples. Many of these examples are new and these indicate the power of the method which does not entail the computation of the tail probabilities for the distribution of the test statistic under a specific alternative. The Bahadur efficiency has also been obtained for all the tests considered in these examples. An example is also provided where the ranks are asymptotically fully informative in Hodges - Lehmann efficiency sense but not so in Bahadur efficiency sense.

2. HODGES - LEHMANN EFFICIENCY:

Broadly speaking, the rate at which the power of a test at a specific alternative tends to one as the sample size tends to infinity is an indication of the performance of the test at the particular alternative. Suppose for definiteness the null hypothesis is a simple one and θ is a particular alternative. For a given level α, let $(1 - \beta_n(\alpha))$ be the power of the test under consideration based on sample size n. In many situations, $\beta_n(\alpha) \to 0$ as $n \to \infty$ exponentially fast. i.e. there exists a parametric function $d(\theta)$ (independent of α) defined on the alternative set with $o < d < \infty$ such that

$$n^{-1} \log \beta_n(\alpha) \to -\frac{d(\theta)}{2} \quad \text{as} \quad n \to \infty.$$

$d(\theta)$ is a measure of the efficiency of the test and has been called by Hodges and Lehmann as the index of the test. For the Bahadur efficiency, one fixes the power at a specific alternative θ as β, $o < \beta < 1$ and considers the exponential convergence to zero of the size α_n of the test.

Bahadur and Raghavachari [3, Theorem 4] proved a theorem and indicated that the dual of the theorem can be used to compute Hodges - Lehmann efficiency. We state this theorem which gives a scheme of computing Hodges - Lehmann efficiency. The proof of the theorem is omitted since it parallels that of Theorem 4 of Bahadur and Raghavachari [3].

Consider a space S of points s, a σ-field $\mathcal{A}$ of sets of S. Let $\{\mathcal{B}_n : n = 1, 2, \ldots\}$ be a sequence of σ-fields such that

$$\mathcal{B}_n \subset \mathcal{A} \qquad n = 1, 2, \ldots\ldots\ldots$$

Let P_o and P_θ be probability measures on $\mathcal{A}$. We assume that P_o and P_θ are mutually absolutely continuous on $\mathcal{B}_n$, and let

$$d\,P_o = r_n(s)\, d\,P_\theta \quad \text{on } \mathcal{B}_n$$

where r_n is $\mathcal{B}_n$ - measurable, $o < r_n < \infty$, $n = 1, 2, \ldots\ldots$ Let

$$K_n^{(1)}(s) = n^{-1} \log r_n(s)$$

Let α be given, $o < \alpha < 1$ and for each n, let $1 - \beta_n = 1 - \beta_n(\alpha)$ be the power of test when P_θ obtains based on r_n with level α.

<u>Theorem 1:</u>

Suppose that $K_n^{(1)}(s) \to K^{(1)}$ as $n \to \infty$ a.o. P_o (or in P_o- probability) where $K^{(1)}$ is a constant, $o < K^{(1)} < \infty$. Then for each α, $n^{-1} \log \beta_n(\alpha) \to -K^{(1)}$ as $n \to \infty$.

3. <u>EXAMPLES:</u>

In the following examples $\mathcal{B}_n$ is the σ-field induced by $T_n(s)$, the test statistic based on independent observations

$x_1, \ldots\ldots x_n$. The simple null hypothesis is indicated by $\theta = \theta_o$ and a specific alternative by θ. P_o refers to the null hypothesis and P_θ to the specific alternative under consideration.

Example 1: $x_1, x_2, \ldots\ldots x_n$ are independent and identically normally distributed variables with mean θ and variance σ^2 (known). Consider testing $H_o : \theta = \theta_o = 0$ against $H_1 : \theta > 0$. Take $T_n = \bar{x}$ and take a specific $\theta > 0$. Here $n^{-1} \log r_n(s) \to \theta^2/2\sigma^2$ a.e. P_o and the Hodges Lehmann index is θ^2/σ^2. The index for this case was originally obtained by Hodges and Lehmann [4] by evaluating tail probabilities.

Example 2: If one takes in Example 1, $T_n = (\# \text{ of } x_i > 0)$, the test based on T_n is the sign test. In this case, $n^{-1} \log r_n(s) \to -\frac{1}{2} \log (4pq)$ a.e. P_o where $P_\theta(X_i > 0) = p$. The index is therefore $-\log (4pq)$. This was again originally obtained by Hodges and Lehmann [4] by obtaining estimates of the tail probabilities of the binomial distribution.

Example 3: $x_1, x_2, \ldots\ldots x_n$ are independent and identically distributed as Poisson with parameter θ. Suppose that $H_o : \theta = \theta_o$ and $H_1 : \theta > \theta_o$. Take $T_n = x_1 + x_2 + \ldots + x_n$.

Then $n^{-1} \log r_n(s) \to \theta - \theta_o + \theta_o \; (\log \theta_o - \log \theta)$ a.e. P_o. The index is therefore $2(\theta - \theta_o) + 2\theta_o(\log\theta_o - \log\theta)$. The Bahadur exact slope turns out to be $2(\theta_o - \theta) + 2\theta(\log\theta - \log\theta_o)$.

<u>Example 4:</u> Consider testing $H_o : \theta = \theta_o$ against $H_1 : \theta < \theta_o$ in the negative binomial case,

$$P\{T_n = t\} = \binom{n+t-1}{n-1} \theta^n q^t, \quad t = 0, 1, \ldots\ldots \quad q = 1 - \theta$$

The index is verified to be $-2\left\{ \log\left(\frac{\theta}{\theta_o}\right) + \left(\frac{q_o}{\theta_o}\right) \log\left(\frac{q}{q_o}\right)\right\}$.

The Bahadur exact slope is given by

$$2\left\{ \log\left(\frac{\theta}{\theta_o}\right) + \left(\frac{q}{\theta}\right) \log\left(\frac{q}{q_o}\right)\right\}.$$

<u>Example 5:</u> $x_1, x_2, \ldots. x_n$ are independent and identically normally distributed with mean ξ and variance θ. Consider testing $H_o : \theta = \theta_o$ against $H_1 : \theta > \theta_o$. Take $T_n = \sum_{i=1}^{n} (x_i - \bar{x})^2$. It can be verified that the index is $[\log\left(\frac{\theta}{\theta_o}\right) - 1 + \frac{\theta_o}{\theta}]$. The exact slope is $[\log\left(\frac{\theta_o}{\theta}\right) - 1 + \frac{\theta}{\theta_o}]$. The exact slope was also obtained in G-Sievers [8] using different methods.

Example 6: $x_1, x_2, \ldots\ldots x_m$ are independent and identically normally distributed with mean ξ and variance σ_1^2. $y_1, y_2, \ldots\ldots y_n$ are independent and identically normally distributed with mean η and variance σ_2^2. Let $\theta = \sigma_1^2 / \sigma_2^2$. Consider $H_o : \theta = \theta_o$ against $H_1 : \theta > \theta_o$. Take

$T_{m,n} = S_1^2/S_2^2$ where $S_1^2 = \sum_{i=1}^{m} (x_i - \bar{x})^2 / (m-1)$ and

$S_2^2 = \sum_{j=1}^{n} (y_j - \bar{y})^2 / (n-1)$. Let $N = m+n$ and let m and n tend to infinity in such a way that $m/N \to \lambda$, $0 < \lambda < 1$. It can be verified that the index is given by

$$\log(\lambda + (1 - \lambda) \frac{\theta_o}{\theta}) - 2(1- \lambda) [\log \theta_o - \log \theta].$$

When $\lambda = 1/2$ (equal sample sizes case), the index reduces to

$$- \log 2 + \log (\theta + \theta_o) - \log \theta_o.$$

The exact slope can be verified to be

$$\log(\lambda + (1- \lambda) \frac{\theta}{\theta_o}) + 2(1- \lambda) [\log \theta_o - \log \theta]$$

Example 7: Let $(x_1, y_1), (x_2, y_2), \ldots (x_n, y_n)$ be independent and identically distributed in the bivariate normal form with correlation coefficient θ, $0 \leq \theta < 1$. Consider testing $H_o : \theta = 0$ against $H_1 : \theta > 0$. Take T_n to be the sample correlation coefficient: r. The density of r when θ is the population correlation coefficient is

$$f_\theta(r) = \frac{(n-2)(1-\theta^2)^{\frac{n-1}{2}}(1-r^2)^{\frac{n-4}{2}}}{\pi(1-\theta r)^{\frac{2n-3}{2}}} \int_0^1 \frac{(1-v)^{n-2}}{(2v)^{1/2}} \{1-\tfrac{1}{2}v(1+\theta r)\}^{-\frac{1}{2}} dv.$$

Observe that for $-1 \leq r \leq 1$ and $0 \leq v \leq 1$,

$$A < [1 - \tfrac{1}{2} v(1+\theta r)]^{-1/2} < B$$

$$\text{and } C < (1 - \tfrac{1}{2} v)^{-\frac{1}{2}} < D$$

with A, B, C and D are constants independent of n.

It can then be verified that

$$\lim_{n\to\infty} n^{-1} \log r_n(s) = \lim_{n\to\infty} n^{-1} \log \left[\frac{(1-\theta r)^{\frac{2n-3}{2}}}{(1-\theta^2)^{\frac{n-1}{2}}} \right]$$

a.e. when $\theta \geq 0$ obtains. Thus the index in this case is $-\log(1-\theta^2)$. It is also the exact slope in this situation.

<u>Example 8:</u> Consider testing $H_0 : \theta = 0$ against $H_1 : \theta > 0$ on the basis of a sample of size n from a p-variate normal distribution ($p \geq 3$) with θ as the multiple correlation coefficient. Take $T_n = R$, the corresponding sample multiple correlation coefficient. It is known (see for example Lehmann [6] p. 320 or Simaika [9]) that among all tests based on R, the uniformly most powerful test rejects H_0 for large value of R. The density of R^2 when the population parameter is θ is proportional to

$$g_\theta(R) = (1-\theta^2)^{\frac{n-1}{2}} (1-R^2)^{\frac{n-p-2}{2}} \int_{-R}^{R} (R^2-t^2)^{\frac{p-4}{2}} \left[\int_0^\infty \frac{d\beta}{(\operatorname{Cosh}\beta - \theta t)^{n-1}} \right] dt \qquad \ldots (3.1)$$

See Kendall and Stuart [5, p. 339].

By making the transformation $\operatorname{sech}\beta = (1-v)/(1-\theta tv)$ and simplifying, (3.1) equals

$$(1-\theta^2)^{\frac{n-1}{2}} (1-R^2)^{\frac{n-p-2}{2}} \int_{-R}^{R} \frac{(R^2-t^2)^{\frac{p-4}{2}}}{(1-\theta t)^{\frac{2n-3}{2}}} \left(\int_0^1 \frac{(1-v)^{n-2}}{(2v)^{1/2}} (1-\tfrac{1}{2}v(1+\theta t)^{-\frac{1}{2}} dv \right) dt \qquad \ldots (3.2)$$

We have

$$\int_{-R}^{R} \frac{(R^2-t^2)^{\frac{p-4}{2}}}{(1-\theta t)^{\frac{2n-3}{2}}} dt = R^{p-3} \int_0^\pi \frac{\operatorname{Sin}^{p-3}\psi \, d\psi}{(1-\theta R\cos\psi)^{\frac{2n-3}{2}}} \qquad \ldots (3.3)$$

(3.3) can be seen to be true by making the transformation $t = R\cos\psi$ in the integral on the left side of (3.3). The right side of (3.3) is $\leq R^{p-3}(1-\theta R)^{-(2n-3)/2}\pi$. Thus we have after some reduction,

$$\frac{g_\theta(R)}{g_o(R)} \leq C\,(1-\theta^2)^{\frac{n-1}{2}} (1-\theta R)^{-\frac{2n-3}{2}}$$

where C does not depend on n. Since $R \underset{n\to\infty}{\to} \theta$ a.e. when θ obtains,

$$\limsup_{n\to\infty} n^{-1} \log r_n(s) \leq \tfrac{1}{2}\log(1-\theta^2) \text{ a.e. } P_o \qquad \ldots (3.4)$$

Observe now that the integral on the left of (3.3) equals (by making the transformation $t = z R$)

$$R^{p-3} \int_{-1}^{1} \frac{(1-z^2)^{\frac{p-4}{2}}}{(1-\theta zR)^{\frac{2n-3}{2}}} dz \geq R^{p-3} \int_{1-\frac{1}{n}}^{1-\frac{1}{n^2}} \frac{(1-z^2)^{\frac{p-4}{2}}}{(1-\theta zR)^{\frac{2n-3}{2}}} d z \quad \ldots\ldots (3.5)$$

and the right side is

$$\geq R^{p-3} [1 - \theta R(1 - \tfrac{1}{n})]^{-(\frac{2n-3}{2})} (\tfrac{1}{n} - \tfrac{1}{n^2}) [1 - (1 - \tfrac{1}{n^2})]^{\frac{p-4}{2}}$$

After some reduction it can be shown that

$$\liminf_{n \to \infty} n^{-1} \log r_n(s) \geq \tfrac{1}{2} \log (1 - \theta^2) \quad \text{a.e.} \quad P_o \quad \ldots\ldots (3.6)$$

Combining (3.4) and (3.6), we conclude that the index equals $-\frac{1}{2} \log (1 - \theta^2)$. The Bahadur exact slope can also be shown to be equal to $-\frac{1}{2} \log (1 - \theta^2)$. Note that the exact slope and index do not depend on p.

Example 9: (Rank test for one sided distributions):

Let $X_1, X_2, \ldots, X_m$ be independent and identically distributed with continuous distribution function $F(x)$ with $F(\Delta) = 0$, $-\infty < \Delta < \infty$. Let $Y_1, Y_2, \ldots, Y_n$ be independent and identically distributed with c.d.f. $G(x) = F(x - \theta)$. For the test of hypothesis $H_o : \theta = 0$ against the alternative $H_1 : \theta > 0$, consider the non-parametric rank test (V-test) which rejects when $V = S_1$ is too large where S_1 is the rank of the smallest Y in the combined sample of the X's and Y's. This rank test does not seem to have been discussed so far for the location problem. Consider also the Z-test that rejects H_o for large values of $Z = \min(Y_1, \ldots, Y_n) - \min(X_1, \ldots, X_m)$. We shall evaluate the Bahadur efficiency and Hodges-Lehmann efficiency of the S-test relative to Z-test when the underlying distributions are exponential, that is, $F(x) = 1 - \exp(-(x-\Delta))$, $x \geq \Delta$. First we have

Lemma 1: If $F(x) = 1 - \exp(-(x-\Delta))$ for $x \geq \Delta$, then the V-test is uniformly most powerful among all rank tests to test the hypothesis $H_o : \theta = 0$ against the alternatives $H_1 : \theta > 0$.

Proof: We know by Hoeffding s theorem see Lehmann [6], p. 237 that if F and G have densities f and g such that f is positive whenever g is, the joint distribution of $S_1, S_2, \ldots, S_n$ is given by

$$P\{S_1 = s_1; S_2 = s_2; \ldots; S_n = s_n\} = \left[E\left\{\frac{g(V^{(s_1)}) \ldots g(V^{(s_n)})}{f(V^{(s_1)}) \ldots f(V^{(s_n)})}\right\}\right] \Big/ \binom{N}{n}$$

where $X_1, X_2, \ldots X_m$ are independent observations from the distributions with c.d.f. F and $Y_1, Y_2, \ldots, Y_n$ are independent observations from the distribution with c.d.f. G;

$V^{(1)} < V^{(2)} < \dots < V^{(N)}$ is an ordered sample of size $N = m + n$ from the distribution F and $S_1 < S_2 < \dots < S_n$ are the ranks of the Y s in the combined sample of the X's and Y's. Here $g(y) = \exp\{-(y - \theta - \Delta)\}$ for $y \geq \theta + \Delta$ and $= 0$ otherwise, so that we have

$$P_\theta\{S_1 = s_1 ; S_2 = s_2 ; \dots ; S_n = s_n\} = \exp\{n\theta\} \; P\{V^{(S_1)} > \theta + \Delta\} / \binom{N}{n} \quad \dots (3.7)$$

Let us denote by $H(S_1)$ the c.d.f. of $V^{(S_1)}$. The most powerful test for testing $H_0 : \theta = 0$ against the alternative $K : \theta = \theta_1 > 0$ rejects when $S_1 \geq C$ where C does not depend on θ_1. Hence the theorem.

Remark: Lemma 1 can also be deduced from a result of I.R. Savage [7] regarding uniform distributions.
For a given $\theta > 0$, we have from (3.7) that

$$P_\theta\{S_1 = s\} = \exp(n\theta). \binom{N-s}{n-1} P\{W^{(s)} > \theta\} \;/\; \binom{N}{n} \quad \dots (3.8)$$

where $W^{(s)}$ is the s^{th} order statistic of a sample of size N from the exponential distribution with density: $\exp(-x)$, $x \geq 0$. It can be shown (we omit the details) using the relation between the exponential distribution and the uniform distribution that

$$\frac{1}{N} \log[P_\theta(S_1 = s) \;/\; P_0(S_1 = s)] = \left(\frac{n}{N}\right)\theta + \frac{1}{N} \log P\{U^{(N-s+1)} < e^{-\theta}\} \quad \dots (3.9)$$

where $U^{(N-s+1)}$ is the $(N-s+1)^{th}$ order statistic of a sample of size N from the uniform distribution over (0, 1). It is well known that

$$P\{U^{(N-s+1)} < e^{-\theta}\} = I_{\mu}(N-s+1,\ s) = \sum_{j=N-s+1}^{N} \binom{N}{j} \mu^{j}(1-\mu)^{N-j} \quad ..(3.10)$$

where we have written μ for $\exp(-\theta)$. It can be shown that as $N \to \infty$ such that $m/N \to \lambda$, $0 < \lambda < 1$, $S_1/N \to \lambda(1-\exp(-\theta))$ with probability one when θ obtains. It follows from (3.2), p. 332 of Hodges and Lehmann [4] that with $\nu = 1 - \lambda\,(1 - \exp(-\theta))$,

$$\frac{1}{N}\log P\{U^{(N-s+1)} < \mu\} \xrightarrow[n\to\infty]{} r \log[\mu/\nu] + (1-\nu)\log\{(1-\mu)/(1-\nu)\} \quad .... (3.11)$$

with probability one when θ obtains.

Denoting the right side of (3.9) by $M(\theta)$ we have from that

$N^{-1} \log[P_\theta(S_1 = s)/ P_0(S_1 = s)] \to (1 - \lambda)\,\theta + M(\theta)$ with probability one when θ obtains.

We proceed to obtain Hodges-Lehmann index and Bahadur exact slope for the test relative to Z-test. Theorem 1 is not readily applicable here because in the original (x, y) space of observations, P_θ and P_0 are not mutually absolutely continuous. For $\theta > 0$, $P_\theta << P_0$ but $P_0 \not\ll P_\theta$. However note that $N^{-1} \log\,(P_\theta(S_1=s)/P_0(S_1 = s)) \xrightarrow[n\to\infty]{} (1 - \lambda)\,\theta + M$ and

$-\lambda\theta$ a.e. P_θ and P_o respectively. Further since by (3.9) $N^{-1} \log (P_\theta\ (S_1 = s) / P_o\ (S_1=s))$ is strictly increasing in s, the tests based on S_1 and $P_\theta\ (S_1=s)/P_o(S_1 = s)$ are equivalent. A slight change in the proof given in Bahadur [2] p. 316-317 for Stein's lemma shows that the Bahadur exact slope and Hodges-Lehmann index for the V-test are still given by a.e. (P_θ) limit of $N^{-1} \log [P_\theta (S_1=s) / P_o(S_1 = s)]$ and a.e. (P_o) limit of $N^{-1} \log [P_o(S_1 = s) / P_\theta (S_1 = s)]$ respectively.

The exact slope is thus $2[(1-\lambda)\theta + M(\theta)]$. In order to compute the index, note first that $S_1/N \underset{n\to\infty}{\to} 0$ a.e. P_o and from (3.11) $M(\theta) = -\theta$. The index is therefore

$$-2[(1-\lambda)\theta - \theta] = 2\lambda\theta.$$

For the Z-test, denoting by $\phi(\theta)$ the density of Z when θ obtains, it can be verified that $N^{-1} \log [\phi(\theta)/ \phi(\ )]$ $\underset{n\to\infty}{\to} (1-\lambda)\theta$ a.e. θ obtains and $\to -\lambda\theta$ under H_o. The Hodges-Lehmann index for the Z-test is therefore $2\lambda\theta$ and the exact slope is $2(1-\lambda)\theta$. It can be seen therefore that the V-test and Z-test are equivalent in the Hodges-Lehmann efficiency sense but V-test is inferior to Z-test (since $M(\theta) < 0$) in the Bahadur efficiency sense. This shows that for the example under consideration, ranks are not fully informative even asymptotically in the Bahadur efficiency sense while they are fully informative asymptotically in the Hodges-Lehmann efficiency sense.

REFERENCES

1. Bahadur, R.R. (1960). Stochastic Comparison of Tests. Ann. Math. Statist. 31 276-295.

2. Bahadur, R.R. (1967). Rates of Convergence of Estimates and Test Statistics. Ann. Math. Statist., 38 303-324.

3. Bahadur, R.R. and Raghavachari, M. (1971). Some asymptotic properties of likelihood ratios on general sample spaces. Proc. Sixth Berkeley Symp. Math. Statist. and Prob. 1, 129-152.

4. Hodges, J.L., Jr. and Lehmann, E.L. (1956). The efficiency of some non-parametric competitors of the t-test. Ann. Math. Statist. 27 324-335.

5. Kendall, M.G. and Stuart, A. (1967). The Advanced Theory of Statistics Vol.2. Second Edition. Hafner Publishing Company, New York.

6. Lehmann, E.L. (1959). Testing Statistical Hypotheses. John Wiley and Sons, Inc., New York.

7. Savage, I.R. (1956). Contributions to the theory of rank order statistics: two sample case. Ann. Math. Statist. 27 590-615.

8. Sievers, G.L. (1969). On the probability of large deviations and exact slopes. Ann. Math. Statist. 40 1908-1921.

9. Simaika, J.B. (1941). An optimum property of two statistical tests. Biometrika 32 70-80.

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