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ELICITATION OF SUBJECTIVE PROBABILITIES IN THE CONTEXT OF DECISION-MAKING

by

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Abstract

In this paper a framework is developed to seek information from the decision-maker about uncertain events, and to use this information for identifying a preferred alternative. Existing methods are examined critically to determine the best strategy for eliciting subjective probabilities in the context of decision-making. It is shown how a preferred alternative can be identified with less than perfect knowledge of the probabilities.

In this paper a framework is developed for eliciting the subjective probabilities of the decision-maker. Elicitation of the subjective probabilities is needed for several decision models e.g. risk analysis, decision-matrix models, decision trees, monte carlo simulation, PERT, etc. (see Huber [12]). The emphasis here is on the decision situations in which a decision alternative has to be chosen from a finite set of alternatives and the outcome of each alternative is contingent on some uncertain events. In section 1., an overview of the theory of subjective probability is given. The procedures for eliciting subjective probabilities are examined in section 2. The procedure most suitable for eliciting probabilities in the context of decision-making is discussed in section 3. In section 4., we develop a framework for identifying a preferred alternative when the exact estimates of probabilities are difficult to obtain. It is shown that a preferred alternative can be identified with simpler judgments from the decision-maker than are necessary for specifying the point estimates of the probabilities of uncertain events. In section 5, a summary of this paper is provided.

1. Overview of the Theory of Subjective Probability

Probabilities have different interpretations, but the view adopted here is the one common to the theory of subjective probability. For an axiomatic development of this theory see Savage [23], or de Finetti [4] [6]. Fishburn [10] also gives an axiomization leading to the definition of subjective probability. For the development of subjective probability, Kyburg and Smokler [13] provide an excellent collection of articles. In a recent paper, de Finetti [7] emphasizes that "subjective probability is the only meaningful interpretation of the word 'probability'".

The subjective probability of an event is the degree of belief (faith or confidence) that a person has in the occurrence of the event on the basis of available information. Thus, the subjective probability for the same event could differ from person to person, and could change for the same person as his pool of available information alters with the passage of time or otherwise. The subjective probabilities of some real world events may be the same for different people (objective probabilities. Marschak [14] states that this would be the case when the events fulfill certain symmetry requirements. For a discussion of the relationship between subjective and objective probabilities see Marschak [15].

Ramsey [20] was the first author to state explicitly that the examination of the overt behavior of a person is the only sound way to measure his subjective probability. However, the measurement process is far from trivial. Marschak and Radner [16, Chapter 1] describe a general method for deriving subjective probabilities from preference ordering on alternatives. We shall discuss such methods in section 2.2.

Any method to examine the choice behavior of a person for the purpose of eliciting subjective probabilities should look for two properties, consistency and coherence. Consistency checks the logical relationship among various choices or judgments of a person so that they do not contradict one another. Inconsistency may arise because of lack of training, education, knowledge or patience. But inconsistency is easy to check, and the person can usually be convinced to modify his judgments as easily as an accountant can be convinced of a numerical error. However, only the decision-maker can modify his own judgments unlike the latter case, when anyone knowledgable about the rules of arithmetic and the principles of accounting can correct the error.

Coherence means that there is a one to one correspondence between a person's revealed choices or judgements and his true feelings or beliefs. Coherence is impossible to check, but as will be seen later, a person can be induced to make coherent choices. Subjectivistic theory requires coherence for subjective probability to be meaningful, but it is a logical requirement and the <u>only</u> requirement of this theory. Coherence refers to internal consistency, and Ramsey [20], de Finetti [4] point out that a person making coherent choices would not select the bets in which he surely loses, regardless of which uncertain event occurs.

2. <u>Review of the Procedures for Eliciting Subjective Probabilities</u>

2.1. <u>Scoring Rules</u>

Over the years several ways have been suggested that induce or encourage a person to make choices (assess subjective probabilities) which are coherent (consistent with his beliefs). A major portion of the literature is devoted to the theoretical investigation of a class of devices called scoring rules (penalty functions) which reward a person if his choices are coherent, and penalize him otherwise. The conceptual base for these rules is the assumption that a rational man would reveal his true beliefs if any departure from it would result in diminution of his reward. Scoring rules are generally employed to discourage intentional bias on the part of a probability assessor who is providing probabilities for someone else's decision-making (e.g., a weatherman). However, in section 3., we show that the scoring rules in conjunction with some other methods can be used for helping an individual assess his probabilities for his own decisions.

de Finetti [3] [5] has discussed scoring rules in detail. Savage [24] provides an excellent study of scoring rules. Following

Savage, if a person associates a subjective probability p with the occurrence of an event E, and if he gets a reward Y (x) if E occurs and Z (x) if E does not occur, where x is the number that he chooses, then the functions Y and Z should be so chosen that it will be in the interest of the subject to specify x equal to p.

In the above scheme it is assumed that the person wants to maximize his expected utility, and that the utility of monetary reward is linear. To avoid the problems of the nonlinear utility of monetary rewards, the monetary rewards can be kept small, but large enough to keep the subject interested. Raiffa [19] and Winkler [32] discuss scoring rules when utility functions are nonlinear.

Three well known simple scoring rules are: the Quadratic scoring rule (Brier [2], de Finetti [3] [5]), the Spherical scoring rule (Toda [28], Roby [21], and the Logarithmic scoring rule (Good [11], Van Naerssen [29]). Many scoring rules can be generated if a general rule is followed, as discussed by Savage [24].

Scoring rules have been used in metereology (Murphy [17], Stael von Holstein [26], Winkler and Murphy [34] [35], educational testing (de Finetti [5], stock market prices (Stael von Holstein [27], and European and American football (de Finetti [3], Winkler [33]).

2.2. Betting Methods

In betting methods (e.g., see Raiffa [18], Schlaifer [25]), to elicit subjective probabilities, a subject is asked to choose from two bets (gambles or lotteries). One bet is contingent on the event whose probability is to be estimated, and the other on a device with well known outcome probabilities (dice, a roulette wheel). To obtain the probability of an event, the following methods can be used.

Method 1: offer the subject the following two bets:

bet 1: Win a reward R if the event E_1 occurs; otherwise nothing.

bet 2: Win a reward R with known probability q; otherwise nothing.

If the subject chooses the bet 1, then $p_1 \geqq q$; otherwise, $p_1 \leqq q$, where p_1 is the subjective probability of the event E_1.

Method 2: offer the subject the following two bets:

bet 1: Win \$R if E_1 occurs; lose \$L otherwise.

bet 2: Win \$L if E_1 does not occur; lose \$R otherwise.

If the subject chooses the first bet then,

$$Rp1 - L\ (1-p_1) \geqq L(1-p_1) - Rp1, \text{ or}$$

$$p_1 \geqq L/(L + R).$$

It should be noted that the method 1 is utility free. Thus, the choice of R does not affect the subjective probability. Both methods result in inequalities on the values of p_1. By asking several questions (bets), the probability of the event can be estimated with sufficient precision.

2.3 Direct Interrogation

Direct interrogation of a subject to elicit his subjective probabilities is by far the most common procedure in most real life studies. Huber [12] reviews various direct interrogation methods and cites numerous empirical studies. Savage [23 p. 27], along with several others, has criticized this method on the grounds that verbal responses may not correspond with a person's behavior in the face of uncertainty. The only reason for its popularity lies in its simplicity, or the complexity of alternative theoretically sound approaches. Huber [12] suggests that betting methods require complex information processing, and thus should be avoided. Winkler [30] suggests that direct interrogation in conjunction with betting methods (using betting methods or scoring rules to check the internal consistency) is a viable alternative. Winkler [31] has used various direct interrogation methods in the experiments to assess subjective probability distributions.

3. Procedure for Eliciting Subjective Probabilities in the Context of Decision-making

In this section we shall examine which method or combination of methods are suitable for the elicitation of subjective probabilities in the context of decision-making.

There are two distinct strategies to elicit subjective probabilities The first strategy is to seek verbal statement of probabilities from the decision-maker. In this strategy, the decision-maker himself translates his feelings of uncertainty into probability numbers. Both the direct interrogation methods, and the scoring methods employ this strategy. The second strategy is to pose some simple choice situations to the decision-maker and infer his judgmental probabilities from his choices. Betting methods employ this strategy.

Because we are concerned with making decisions in the presence of uncertainty, we should infer a decision-maker's feelings of uncertainty as these influence his choices. Therefore, in the context of decision-making, the second strategy is a better way to elicit subjective probabilities. However, if for some reason the first strategy is to be used, then the decision-maker must be trained to understand as well as possible the necessary correspondence between his own beliefs and the numerical probabilities into which they must be translated (see de Finetti [5, pp.89-90]).

In many decision situations, however, a combination of methods employing the second strategy may be more appropriate than any single method alone. Below, an example is given to demonstrate this point of view.

Example

In a centrally planned economy like India, foreign exchange allocations to various industrial sectors are decided by a Planning Commission. In a simplified situation, assume that these allocations depend on how much foreign exchange would be needed to import food grains. If rainfall is bad, a major portion of the foreign exchange is to be kept aside. Thus, to arrive at an optimal allocation plan, the planning commission needs the subjective probability of rainfall (good or bad) of the policy-maker, say the Planning Minister. We examine different procedures to elicit the subjective probability of the Planning Minister.

Direct Interrogation

Analyst (A) : In your opinion which is more likely, good rainfall or bad rainfall?

Planning Minister (P.M.): I suppose good rainfall.

A : What would you estimate as the chance of good rainfall?

P.M.: I do not know. Why don't you talk to a meteorologist?

Scoring Method (After explaining the scoring rule, its implications etc.)

A : Now, what number between 0 and 1 do you choose for good rainfall. Remember you have to maximize your Rs. (Indian currency) gain. And, money will actually change hands once we have observed the events.

P.M.: Who is your boss? Your salary may change hands after this event!

Betting Method (Monetary reward)

A : Sir, choose one of the following two lotteries:

lottery 1: If good rainfall occurs you will get Rs.1000; otherwise nothing.

lottery 2: If in a toss of a coin, a "head" comes up, you will get Rs.1000; otherwise nothing.

P.M.: Why don't you play this game with my son? He likes gambling anyway.

Betting Method (Non-monetary reward, hypothetical gambles)

Analyst discusses with the Planning Minister the implications of rainfall on grain output. The Planning Minister already knows a lot about it, and they agree that good rainfall would result in 100 million tons (MT) of grain output, whereas, only 95 MT would be produced if rainfall is bad.

A: Sir, would you choose a plan in which 100 MT of grain would be produced if rainfall is good, and 95 MT would be produced if rainfall is bad; or a plan in which, irrespective of rainfall, there is a 50% chance of producing 100 MT and 50% chance of producing 95 MT.

P.M.: What is the second plan?

A: Sir, it is a hypothetical plan.
(A briefs the P.M. on the purpose of such hypothetical questions)

P.M.: I would choose the first plan.

A: Sir, would you still choose the first plan if in the second plan, the chance of producing 100 MT is 60%?

P.M.: I never was too good in statistics at Oxford. But, if you still want my opinion...........

A: (wonders whether P.M. missed the point).

<u>A Good Method</u> (no hypothetical gambles and no monetary rewards)

A: Sir, which one of the following two plans would you choose (the output of each plan under the two possible events is shown)?

	Good Rainfall	Bad Rainfall
Plan 1	100 MT	95 MT
Plan 2	98.75 MT	98.75 MT

P.M.: Plan 2.

A: Now sir, if I replace Plan 1 by the following plan what would be your choice?

	Good Rainfall	Bad Rainfall
Plan 1 (modified)	99.69 MT	97.19 MT

P.M.: I would now choose Plan 1.

A: Sir, could you please give your choice one more time between the Plan 1 (modified) and the following plan:

	Good Rainfall	Bad Rainfall
Plan 2 (modified)	99.29 MT	98.04 MT

P.M.: I would still choose Plan 1. It looks good.

A: So sir, among the following plans which we considered just now, you would rate Plan 1 (modified) as the best.

	Good Rainfall	Bad Rainfall
Plan 1	100 MT	95 MT
Plan 2	98.75 MT	98.75 MT
Plan 1 (modified)	99.69 MT	97.19 MT
Plan 2 (modified)	99.29 MT	98.04 MT

P.M.: I certainly think so.

It can be seen that in three judgments we have established that the subjective probability of P.M. of good rainfall is between .69 and .75

(see the Appendix 1). We employed a combination of a betting method, a scoring method, and an interval bi-section direct interrogation technique to elicit the subjective probability of the P.M. This example illustrates that an analyst should use the combination of methods which makes the most sense in a specific situation. It can also be seen that successively more difficult judgments are needed in specifying the probabilities with precision.

In the next section we show that in many decision situations such precision is not required in identifying a preferred alternative. From now on in our discussion we shall assume that subjective probabilities are elicited using an appropriate procedure.

4. A Framework for Identifying a Preferred Alternative with Incomplete Knowledge of Probabilities

In this section we develop a framework for identifying a preferred alternative when the exact estimates of the probabilities are difficult to obtain. In section 4.1., the problem statement is given. In section 4.2., decision-making model 1 is considered. In this model it is assumed that the probabilities of uncertain events are not influenced by the alternative chosen. In section 4.3., the results of section 4.2. are generalized to the situations in which the probabilities of uncertain events are influenced by the alternative chosen.

4.1. Problem Statement

A single decision-maker wants to select an alternative from a finite set S of N available alternatives. The outcomes of the alternatives are contingent on the occurrence of one of the uncertain events E_j from m possible events $E_1, E_2, \ldots, E_m$. The decision-maker does not know which of the uncertain events will occur. However, all N x m outcomes, in which O_j^k is the outcome that occurs if the decision-maker selects alternative k and the event E_j occurs, and the utilities of these outcomes, $f(O_j^k)$, can be accurately specified[1].

We shall let p_j^k denote the probability of the occurrence of the event E_j when alternative k is implemented. The probability distribution $(p_1^k, p_2^k, \ldots p_m^k)$ is elicited from the decision-maker. We shall also let F(k) be the expected utility of alternative k. Then,

$$F(k) = \sum_{j=1}^{m} p_j^k \cdot f(O_j^k). \qquad (1)$$

The decision-maker desires to choose an alternative $k^* \in S$ such that $F(k^*) \geq F(k)$ for all k S.

4.2. Decision-making Model I

In the decision-making model I, it is assumed that the probabilities of occurrence of the random events are independent of the alternative selected. Thus, in this model $p_j^k = p_j$ for $k = 1$ to N.

4.2.1. The Two or Three Event Situation ($1 \neq m \leq 3$)

In the two or three event situation, the regions in probability space in which various alternatives are preferred can be easily identified. For example, if $m = 2$, the value of p_1 at which $F(g) = F(k)$ for any $g,k \in S$ is given by solving $p_1 f(O_1^g) + (1-p_1) f(O_2^g) = p_1 f(O_1^k) + (1-p_1) f(O_2^k)$.

$$p_1 = (f(O_2^k) - f(O_2^g)) / (f(O_1^g) - f(O_2^g)) - (f(O_1^k) - f(O_2^k)). \qquad (2)$$

Thus, between the interval p_1 and 1 alternative g is preferred, and, between the interval 0 and p_1 alternative k is preferred.

Similarly, the three event situation, an alternative g is preferred if $F(g) \geq F(k)$ for $k \in S$. Thus, by plotting a line $\sum_{j=1}^{m} p_j \; f(O_j^g) = \sum_{j=1}^{m} p_j \; f(O_j^k)$, we can identify the regions in which alternative g or k is preferred.

Once the regions in which the various alternatives are preferred have been identified, the analyst can frame meaningful questions to seek judgments from the decision-maker. The unnecessary information is thus avoided, and the decision-maker is asked to make choices only to the extent necessary for identifying a preferred alternative. These choices may be much simpler than are required to estimate the probabilities with precision. Below, a simple example is given.

In Table 1, the utilities of the outcomes of the three alternatives under each of the three random events are given.

Suppose the decision-maker reveals $p_1 \geq p_2 \geq p_3$; then the regions in which each of the three alternatives are preferred can be plotted (see Figure 1).

It can be easily seen that exact estimates of the probabilities are not required. For example, if the decision-maker reveals $p_1 \geq 7/9$ then alternative 1 is preferred, similarly if $p_1 \leq 13/30$, alternative 3 is preferred. For $13/30 < p_1 < 7/9$ the relative magnitudes of p_2 and p_3 have to be considered. For example, if $p_2 \geq 13/30$ then alternative 2 is preferred.

4.2.2. The Multiple Event Situation (m>3)

For m >3, complex information processing task is required to select a preferred alternative if the approach of Section 4.2.1. is employed. This is because, for a given range of values for say p_1, an alternative is preferred depending on the relative magnitudes of other p_j values. Thus, too many combinations of the values of the different p_j need to be examined, and then arranged in an orderly fashion so that meaningful questions can be asked to the decision-maker. We therefore prpose a framework in which information processing is simplified considerably, but the decision-maker may be required to render a few more judgments.

In the suggested approach the decision-maker is asked to provide some information about his subjective probability distribution. Based on this information tests are performed to determine whether some alternatives can be excluded. If a preferred alternative is not identified, more information is sought. Specifically, suppose the decision-maker's choices reveal the following ordinal ranking on the probabilities:

$$p_1 \geq p_2 \geq \cdots\cdots \geq p_m.$$

Now, if we show that $F(g) \geq F(k)$ for all probability vectors $(p_1, p_2, \ldots\ldots, p_m)$ satisfying the ordinal ranking and the requirement $\sum_{j=1}^{m} p_j = 1$, then alternative k can be excluded. $F(g) \geq F(k)$ if $Z^*_{gk} \geq 0$, where

$$Z^*_{gk} = \text{Min} \sum_{j=1}^{m} p_j (f(O_j^g) - f(O_j^k)) \text{ s.t. } p_j \geq p_{j+1} \text{ for } j=1 \text{ to } m-1, \tag{3}$$

$$\sum_{j=1}^{m} p_j = 1,\ 0 \leq p_j \leq 1 \text{ for } j = 1 \text{ to } m.$$

The m feasible extreme point solutions of the linear program (3) are: (1,0,.....,0), (1/2, 1/2, 0,....0), ..., (1/m, 1/m,...1/m) (see Appendix 2).

Therefore, $Z^*_{gk} \gtreqless 0$ if the following holds:

$$\sum_{j=1}^{t} f(0^g_j) \geq \sum_{j=1}^{t} f(0^k_j) \text{ for } t = 1 \text{ to } m . \qquad (4)$$

Note that the above result is similar to that in Fishburn [8] [9]. Fishburn has discussed dominance with several measures on the probabilities. Abramson [1] studies the dominance with inequality sets on the probabilities. However, the framework of linear programming is much more general and flexible in incorporating information sought from the decision-maker. For example, suppose the decision-maker can provide only interval estimates on p_j, $\underline{p}_j \leq p_j \leq \bar{p}_j$; the following linear program needs to be solved:

$$Z^*_{gk} = \text{Min} \sum_{j=1}^{m} p_j \ (f(0^g_j) - f(0^k_j)) \text{ s.t. } \underline{p}_j \leq p_j \leq \bar{p}_j \quad \text{for } j=1 \text{ to } m,$$

$$\sum_{j=1}^{m} p_j = 1, \quad 0 \leq p_j \leq 1 \quad \text{for } j=1 \text{ to } m. \qquad (5)$$

If $Z^*_{gk} \geq 0$, then alternative g is preferred to alternative k. Notice that N(N-1) versions of (5) need to be solved for all possible pairs, g,k S. Fortunately, these linear programs can be solved by inspection since (5) is simple "knapsack" problem.

If the ordinal ranking of probabilities is used in conjunction with the interval estimates, or some other information (ranking of first differences, etc.) is incorporated, then the above linear program may have to be solved with standard algorithms. However, the solution of such small linear programs by computer is efficient and inexpensive.

If a preferred alternative is not identified, and if the decision-maker does not wish to reduce further the uncertainty, by tightening the bounds on p_j's, one of the following secondary criteria can be adopted:

1. Choose the alternative with the maximum of the maximum (max max) or with the maximum of the minimum (max min) expected utility. In "max max" strategy the alternative with $\underset{k \ S}{\text{Max}} \ \bar{Z}^k$ is selected where

$$\bar{Z}^k = \text{Max} \sum_{j=1}^{m} p_j \ f(0^k_j) \text{ s.t. } \underline{p}_j \leq p_j \leq \bar{p}_j; \quad 0 \leq p_j \leq 1, \text{ for}$$

In "max min" strategy the above linear program is minimized to obtain the value of $\underline{Z}^k$, and the alternative with the maximum $\underline{Z}^k$ value is selected.

2. Roy [22, p.263] suggests that if themax min strategy is used, the associated probability values should also be presented to the decision-maker. If the decision-maker is not satisfied with some values, the appropriate constraints can be used to exclude such values, and then the process can be repeated using the max min criterion.

3. Present to the decision-maker a table of the maximum value, the minimum value and the associated probability values for each alternative. If the number of alternatives is large, the alternatives with relatively low minimum and low maximum values can be excluded. The decision-maker can choose from the remaining "good" alternatives.

4.3. Decision-making Model II

In the decision-making model II the possibility is allowed that the selection of an alternative may influence the probability of the occurrence of the random events. Decision-making model I is thus a special case of this model. The framework developed in section 4.2. is also applicable in this situation. For example, if the interval estimates of probabilities are available, then $g \succsim k$ if $Z^*_{gk} \geqq 0$, where Z^*_{gk} is given by:

$$Z^*_{gk} = \text{Min} \sum_{j=1}^{m} (p^g_j\ f(O^g_j) - p^k_j\ f(O^k_j))$$

$$\text{s.t.} \quad \sum_{j=1}^{m} p^g_j = 1,\ \underline{p}^g_j \leqq p_j \leqq \bar{p}^g_j \text{ for } j=1 \text{ to } m,\ 0 \leqq p_j \leqq 1 \text{ for } j=1 \text{ to } m,$$

$$\sum_{j=1}^{m} p^k_j = 1,\ \underline{p}^k_j \leqq p_j \leqq \bar{p}^k_j \quad \text{for } j=1 \text{ to } m,\ 0 \leqq p_j \leqq 1 \text{ for } j=1 \text{ to } m. \qquad (6)$$

The above linear program can be simply decomposed into two linear knapsack problems, one of which is minimized and the other maximized. Here, only

2N knapsack problems need to be solved to do all comparisons, as compared with N(N-1) in the decision-making model I. However, the test in (6) is weaker than in (5) since in this model the probabilities of uncertain events are not the same for each alternative.

Similarly, if only the ordinal ranking of the p_j^k's for each alternative k is available, then g k if the following holds:

$$\underline{Z}^g \geqq \bar{Z}^k, \text{ where}$$

$$\underline{Z}^g = \text{Min} \sum_{j=1}^{m} p_j^g f(O_j^g) \text{ s.t. } p_j^g \geqq p_{j+1}^g \text{ for } j=1 \text{ to } m-1,$$

$$\sum_{j=1}^{m} p_j^g = 1,\ 0 \leqq p_j^g \leqq 1 \text{ for } j=1 \text{ to } m,$$

and

$$\bar{Z}^k = \text{Max} \sum_{j=1}^{m} p_j^k f(O_j^k) \text{ s.t. } p_j^k \geqq p_{j+1}^k \text{ for } j=1 \text{ to } m-1,$$

$$\sum_{j=1}^{m} p_j^k = 1,\ 0 \leqq p_j^k \leqq 1 \text{ for } j=1 \text{ to } m.$$

Using the results in (4) we obtain:

$$\underline{Z}^g = \underset{t}{\text{Min}} \frac{1}{t} \sum_{j=1}^{t} f(O_j^g), \text{ and}$$

$$\bar{Z}^k = \underset{t}{\text{Max}} \frac{1}{t} \sum_{j=1}^{t} f(O_j^k), \quad t=1,2,\ldots,m.$$

Thus, an alternative k can be excluded if thefollowing inequality holds;

$$\underset{t}{\text{Min}} \left[\frac{1}{t} \sum_{j=1}^{t} f(O_j^g) \right] \geqq \underset{t}{\text{Max}} \left[\frac{1}{t} \sum_{j=1}^{t} f(O_j^k) \right], \quad t = 1,2,\ldots m. \tag{7}$$

In general, any additional information can be incorporated in the constraints of (6), and the linear program can be solved. However, whenever possible the special structure of the linear program should be exploited as in (7) above. One more special case is considered. Suppose, after specifying the ordinal ranking of probabilities for each alternative k, the decision-maker is willing to provide the rank order on the probabilities of the same event under two different alternatives. This is given by the relation $p_j^g \geqq p_j^k$ for g, k $\in$S. Now, the inequality in (7) becomes a stronger condition because the additional constraints render some combinations of extreme point solutions infeasible. These infeasible combinations are easy to find.

For example, suppose the utility matrix for a choice situation is given in Table 2.

The ordinal ranking on the probabilities is as follows:

$$p_1^1 \gtrsim p_2^1 \gtrsim p_3^1 \gtrsim p_4^1$$

$$p_1^2 \gtrsim p_2^2 \gtrsim p_3^2 \gtrsim p_4^2$$

$$p_3^3 \gtrsim p_2^3 \gtrsim p_1^3 \gtrsim p_4^3$$

$$p_4^4 \gtrsim p_3^4 \gtrsim p_2^4 \gtrsim p_1^4$$

By using (7) it can be seen that alternatives 3 and 4 are excluded, but nothing can be said about alternatives 1 and 2. Now, suppose that the decision-maker reveals $p_1^1 \gtrsim p_1^2$ and $p_3^2 \gtrsim p_3^1$. In Table 3 it is shown that alternative 1 is now preferred to alternative 2. In Table 3 the extreme point solutions for both alternatives are given, and in feasible combinations of the solutions are marked X. To the left of the inequality $\gtrsim$, the expected utility of alternative 1 is given and the expected utility of alternative 2 is given to the right.

5. Summary

In this paper, existing methods were examined critically to determine the best strategy for eliciting subjective probabilities in the context of decision-making. It was argued that a combination of methods for eliciting subjective probabilities is more appropriate in the context of decision-making.

A framework was developed to seek information from the decision-maker about uncertain events, and to use this information for identifying a preferred alternative. Two situations which frequently occur in decision-making under uncertainty were considered. In the first situation, it is assumed that the probabilities of uncertain events are independent of the decision alternative chosen. In the second situation, the possibility is allowed that the probabilities of uncertain events may be influenced by the chosen alternative. It was shown how a preferred alternative can be identified with less than perfect knowledge of the probabilities.

Table 1

An Outcome Matrix

Alternative/Event	E_1 $f(O_1)$	E_2 $f(O_2)$	E_3 $f(O_3)$
1	6	2	4
2	4	8	3
3	4	4	16

Figure 1

Regions in which an Alternative is Preferred

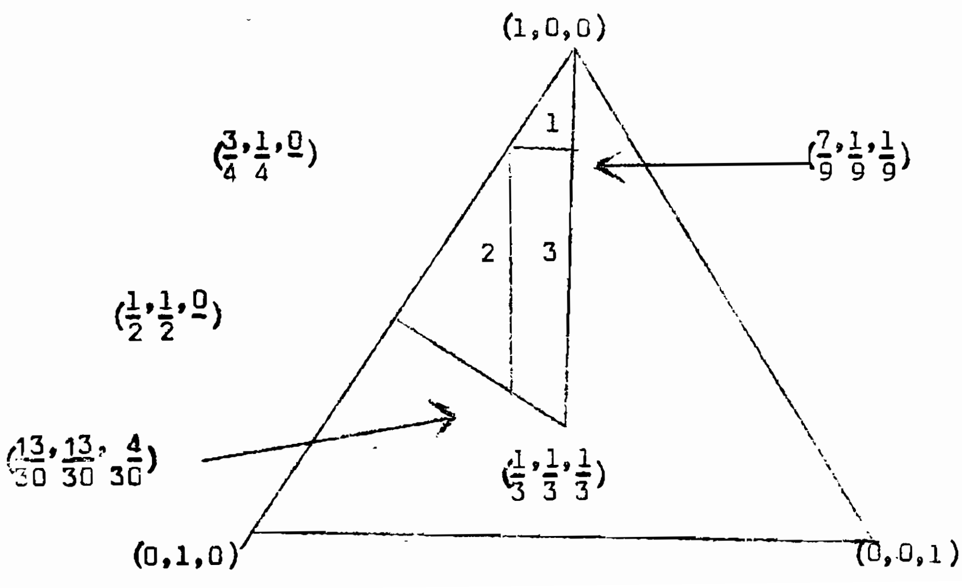

Table 2

An Outcome Matrix for Decision-making Model II

Alternative \ Event	E_1 $f(O_1)$	E_2 $f(O_2)$	E_3 $f(O_3)$	E_4 $f(O_4)$
1	16	8	6	16
2	12	10	8	12
3	8	12	4	2
4	10	12	4	4

Table 3

Feasible Extreme Point Solutions

Alternative 1 Extreme Point \ Alternative 2 Extreme Point	(1,0,0,0) F(1) F(2)	(1/2.1/2,0,0) F(1) F(2)	(1/3.1/3,1/3,0) F(1) F(2)	(1/4,1/4,1/4,1/4) F(1) F(2)
(1,0,0,0)	16 $\geq$ 12	X	X	X
(1/2,1/2,0,0)	16 $\geq$ 11	12 $\geq$ 11	X	X
(1/3,1/3,1/3,0)	16 $\geq$ 10	12 $\geq$ 10	10 $\geq$ 10	X
(1/4,1/4,1/4, 1/4)	16 $\geq$ 10.5	12 $\geq$ 10.5	X	11.5 $\geq$ 10.5

"Footnote"

[1] O_j^k is a single dimensional outcome. However, the results of this section are equally applicable if the outcomes of the alternatives are multi-dimensional, provided the utility function over these outcomes can be accurately specified.

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APPENDIX 1

PROCEDURE FOR ELICITING SUBJECTIVE PROBABILITY OF PLANNING MINISTER

In this Appendix we shall show how we used a combination of procedures in section 3., to elicit the subjective probability of the Planning Minister.

The first question to the Planning Minister was to seek his choice between Plan 1 and Plan 2. The outcomes of the two plans were as follows:

	Good rainfall	Bad rainfall
Plan 1	100	95
Plan 2	98.75	98.75

The Planning Minister preferred Plan 2 and therefore the expected utility of plan 2 is greater than the expected utility of plan 1*.

(1) $98.75\, p_1 + 98.75\, (1-p_1) \geqq 100\, p_1 + 95\, (1-p_1)$,

where p is the subjective probability of good rainfall. Solving (1), we obtain $p_1 \leqq .75$.

In the second question we asked the Planning Minister to choose from Plan 2 and Plan 1 (modified).

	Good rainfall	Bad rainfall
Plan 1 (modified)	99.69	97.19

His choice of Plan 1 (modified) revealed that

(2) $99.69\, p_1 + 97.19\, (1-p_1) \geqq 98.75\, p_1 + 98.75\, (1-p_1)$.

Solving (2) we obtain $p_1 \geqq .625$. Thus, we know that $.625 \leqq p_1 \leqq .75$.

In the last question the Planning Minister preferred Plan 1 (modified) over Plan 2 (modified).

	Good rainfall	Bad rainfall
Plan 2 (modified)	99.29	98.04

Hence,

(3) $99.69\, p_1 + 97.19\, (1-p_1) \geqq 99.29\, p_1 + 98.04\, (1-p_1)$

The solution of (3) gives $p_1 \geqq .69$. Therefore, the subjective probability of the Planning Minister is between .69 and .75.

Notice that the inequalities on p_1 are obtained using betting method 2 (section 2). Also notice that in each successive paired comparison, the interval in which p_1 lies was halved. This is analogous to the interval bi-section scheme in direct interrogation methods.

The four hypothetical plans in our example were derived by using a quadratic scoring rule. In scoring methods, a subject is asked to specify his subjective probability of an event E_1, say p_1. If a quadratic scoring rule is used in rewarding the subject, then his score in case E_1 occurs, $R_1(p_1)$, and his score in case E_1 does not occur, $R_2(p_1)$, are computed as follows:

$$R_1(p_1) = 2p_1 - p_1^2 - (1-p_1)^2$$
$$R_2(p_1) = 2(1-p_1) - p_1^2 - (1-p_1)^2$$

In Table 1, the scores corresponding to four values of p_1 are shown. Since a positive linear transformation of these scores also maintains the properties of a scoring rule, the transformation 97.5 + 2.5 x is used, where x is the original score. This transformation is necessary for meaningful interpretation of the rewards. The transformed scores shown in Table 1 represent the outcomes of various plans under the two events, good rainfall and bad rainfall.

TABLE 1

Computation of Rewards for Planning Example

Plan		Score			Transformed Score[1,2]	
		p_1	$R_1(p_1)$	$R_2(p_1)$	$R_1'(p_1)$ (million tons)	$R_2'(p_1)$ (million tons)
Plan 1		1.000	1.0000	-1.0000	100.00	95.00
Plan 2		.500	.5000	.5000	98.75	98.75
Plan 1	(modified)	.750	.8750	-.1250	99.69	97.19
Plan 2	(modified)	.625	.7188	.2188	99.29	98.04

[1] $R_1'(p_1) = 97.5 + 2.5\,R_1(p_1)$, $R_2'(p_1) = 97.5 + 2.5\,R_2(p_1)$.

[2] [redacted]unded to two decimal places.

APPENDIX 2

EXTREME POINT SOLUTIONS OF THE LINEAR PROGRAMMING PROBLEM WITH ORDINAL RANKING ON VARIABLES

In this Appendix we shall prove that one of the m feasible solutions $(1,0,\ldots,0)$, $(1/2,1/2,0,\ldots,0),\ldots,(1/m,1/m,\ldots,1/m)$ provides an optimal solution to the following linear programming problem:

$$(1) \quad \text{Max} \sum_{j=1}^{m} c_j \cdot p_j \quad \text{s.t. } p_j \geq p_{j+1} \text{ for } j=1 \text{ to } m-1,$$

$$\sum_{j=1}^{m} p_j = 1, \quad 0 \leq p_j \leq 1 \text{ for } j=1 \text{ to } m.$$

Proof: Let $p_j = a_j$, $j=1$ to m be <u>any</u> feasible solution of the linear program (1). We show that this solution can be represented as a convex linear combination of the proposed m solutions, and thus at least one of the proposed m solutions is at least as good as $(a_1, a_2, \ldots, a_m)$.

We wish to find $\alpha_j \geq 0$, $j=1$ to m, with $\sum_{j=1}^{m} \alpha_j = 1$, such that

$$(2) \quad \alpha_1 c + \frac{1}{2}\alpha_2 (c_1 + c_2) + \ldots + \frac{1}{m}\alpha_m (c_1 + c_2 + \ldots + c_m) \geq \sum_{j=1}^{m} c_j a_j$$

To demonstrate this, we may show

$$c_1 \left(\alpha_1 + \frac{1}{2}\alpha_2 + \ldots + \frac{1}{m}\alpha_m\right) \geq c_1 a_1$$

$$c_2 \left(\frac{1}{2}\alpha_2 + \ldots + \frac{1}{m}\alpha_m\right) \geq c_2 a_2$$

$$\vdots \qquad\qquad \vdots$$

$$c_m \left(\frac{1}{m}\alpha_m\right) \quad c_m a_m$$

We construct

$$(3) \quad \alpha_m = m\, a_m$$

$$\alpha_{m-1} = (m-1)(a_{m-1} - a_m)$$

$$\alpha_{m-2} = (m-2)(a_{m-2} - a_{m-1})$$

$$\vdots$$

$$\alpha_2 = (a_1 - a_2)$$

In (3) we have $\alpha_j \gtreqless 0$, j=1 to m, since $a_j - a_{j+1} \gtreqless 0$, and $\sum_{j=1}^{m} \alpha_j = \sum_{j=1}^{m} a_j = 1$. Substituting these values of α_j's into the left hand side of (2), we obtain $\sum_{j=1}^{m} C_j a_j$ which is equal to the right hand side of (2).

Hence, any feasible solution can be represented as a convex linear combination of the proposed m solutions, and therefore in order to obtain a solution of the given linear program (1) it is sufficient to examine only the proposed m extreme point solutions.

"Footnote"

*It is assumed that the utility of the Planning Minister is linear in the amount of foodgrain production. This assumption may be appropriate as the range of outcome values (95 MT to 100 MT) is small. However, if the utility is nonlinear than a scheme as in Raiffa[19] or Winkler[32] needs to be used.