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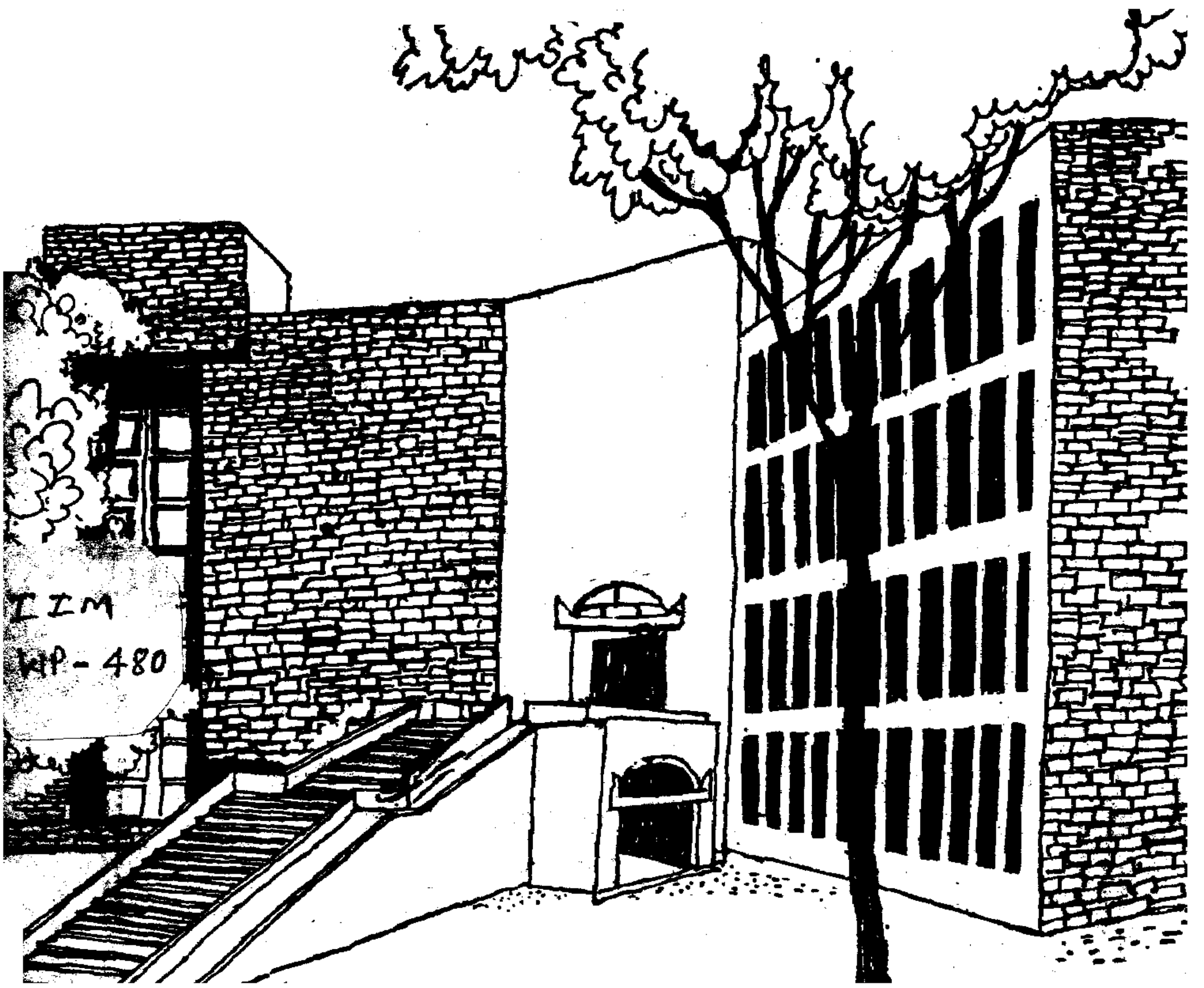

DETERMINISTIC AND RANDOM SINGLE MACHINE SEQUENCING WITH VARIANCE MINIMIZATION

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DETERMINISTIC AND RANDOM SINGLE MACHINE SEQUENCING WITH VARIANCE MINIMIZATION

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In this paper, we discuss the problem of ordering n jobs on a single machine. The objective is to minimize the variance of the completion times of the jobs. The problem of minimizing the variance of completion times or such similar measures has been studied by researchers, e.g. Merten and Muller [5], Schrage [6], Eilon and Chowdhary [2] and Kanet [3, 4].

The paper is divided into two parts. Part I contains the problem of minimizing variance when processing times of jobs are deterministically known. In Part II, we discuss scheduling of jobs when processing times are random variables with known probability distributions. We use the following notations

n : number of jobs to be scheduled.

p_i : processing time of job at position i, in the given sequence.

$p_{(i)}$: i^{th} largest processing time of jobs.

W_i : waiting time for the job at i^{th} position in a given sequence. We have

$$W_i = p_1 + p_2 + \ldots\ldots + p_{i-1}, \qquad i = 1, \ldots\ldots n.$$

C_i : Completion time of the job at i^{th} position in a given sequence. We have

$$C_i = p_1 + p_2 + \ldots\ldots + p_i, \qquad i = 1, \ldots\ldots n$$

$$= W_i + p_i$$

Note that W_i and C_i depend on the sequence of n jobs.

Assumptions:

1. Machine is continuously available for assignment.
2. Preemption is not allowed.
3. All jobs are available at the same time.

With the notations mentioned above, the variance between the completion times is given by

$$V = \frac{1}{n} \sum_{i=1}^{n} (C_i - \bar{C})^2 \qquad (1)$$

where $\bar{C} = (\sum_{i=1}^{n} C_i) / n$. We have from definition of C_i,

$$\bar{C} = \sum_{i=1}^{n} \frac{n - i + 1}{n} p_i \qquad (2)$$

Following Schrage [6], (1) can be written as

$$nV = \sum_{i=1}^{n} \frac{(n-i+1)(i-1)}{2} p_i^2 + \frac{2}{n} \sum_{i=1}^{n-1} (i-1)p_i \sum_{j=i+1}^{n} (n-j+1)p_j \quad (3)$$

Given the processing times of the jobs, the problem is to schedule the jobs so that V is minimum. If the processing times are random variables with known distributions, then the jobs are to be scheduled so that E(V) is minimum, where E stands for expecion.

Merten and Muller [5] showed that, variance of completion times for the sequence

$$R : p_1, p_2, \ldots\ldots p_{n-1}, p_n$$

is the same as the variance of completion times of the sequence

$$R^* : p_1, p_n, p_{n-1}, \ldots\ldots\ldots\ldots p_2.$$

Schrage [5] proved that the job with the largest processing time is at the first position in optimal sequence. Schrage obtained the optimal sequence upto n = 5 jobs and gave a conjecture of the optimal sequence for general n as

$$p_{(1)}, \quad p_{(3)}, p_{(4)}, \quad ?\ldots\ldots\ldots\ldots ?, p_{(2)}.$$

Part of this conjecture, involving the positioning $p_{(4)}$ was proved to be untrue by Kanet [3]. He showed through an example that $p_{(4)}$ can be at (n - 1) position.

Eilon and Chowdhary [2] proved that an optimal sequence is V-shaped and proposed a few heuristic procedures for obtaining sequence for general n. Kanet [3] proposed a heuristic procedure of minimizing

$$TSDC = \sum_{i=1}^{n} \sum_{j=1}^{n} (C_i - C_j)^2$$

which was shown to be equivalent to minimizing variance of completion times.

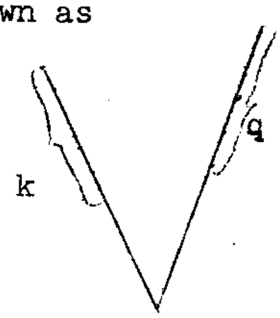
PART-I

Deterministic Case

In this part, we show that for $n \leq 18$ jobs, there exists an optimal solution in which the job with third largest processing time is always at the second position. Using the techniques of partitioning of variance, we obtain a general formula for the change in variance due to the interchange of two jobs whose positions have not been fixed in the sequence. Using the above result, optimal sequences are obtained for n = 6 and n = 7 jobs. A heuristic procedure is given for n jobs. This method is compared with the methods given by Eilon and Chowdhary [2] and Kanet [3].

Variance Partition:

We partition the variance as variance of completion times for fixed positions and variance of completion times for non-fixed positions. To obtain the formula of partitioned variance we use the wellknown formula of pooled variance. Since optimal sequence is V-shaped, we start fixing the positions at the end points of V. Suppose the first k and last q positions are fixed, the positions of the sequence is shown as



This means that 1,2,...., k, n, n-1, ... n-q+1 positions are fixed. Thus the values of completion times in two groups are

Group I: Known Values

$$C_1, C_2, \ldots\ldots, C_k, C_n, \ldots\ldots\ldots C_{n-q}.$$

Group II: Unknown Values

$$C_{k+1}, \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots C_{n-q-1}$$

In the first group there are k+q+1 = m terms and in group II n-m terms. Let

$$\bar{C}_1 = \frac{C_1 + \ldots + C_k + C_{n-q} + \ldots + C_n}{m}$$

$$= \sum_{i=1}^{k+1} \frac{m-i+1}{m} p_i + \frac{q+1}{m} \sum_{i=k+2}^{n-q-1} p_i + \sum_{i=n-q}^{n} \frac{n-i+1}{n} p_i \quad (4)$$

$$\bar{C}_2 = \frac{C_{k+1} + \ldots\ldots\ldots + C_{n-q-1}}{n - m}$$

$$= \sum_{i=1}^{m+1} p_i + \sum_{i=k+2}^{n-q-1} \frac{n - q - i}{n - m} p_i \quad (5)$$

$$\bar{C} = \frac{C_1 + C_2 + \ldots\ldots + C_n}{n}$$

$$= \frac{m \bar{C}_1 + (n-m) \bar{C}_2}{n} \quad (6)$$

Let S_1^2 and S_2^2 be the variances of the first and second group respectively. Then the wellknown formula for pooled variance of two groups is given as

$$nV = mS_1^2 + (n-m)S_2^2 + m (\bar{C}_1 - \bar{C})^2 + (n-m) (\bar{C} - \bar{C})^2$$

$$= mS_1^2 + (n-m) S_2^2 + \frac{m(n-m)}{n} (\bar{C}_1 - \bar{C}_2)^2 \quad (7)$$

In the above expression, S_1^2 is a known quantity. S_2^2 is unknown and does not involve p_{k+1}, and p_{n-q}. From (4) and (5), $(\bar{C}_1 - \bar{C}_2)$ can be simplified as

$$\bar{C}_1 - \bar{C}_2 = \frac{(n-m)Q - Y_m}{m(n-m)} \tag{8}$$

where

$$\left.\begin{aligned} Q &= \sum_{i=n-q}^{n} (n-i+1)p_i - \sum_{i=1}^{n} (i-1)p_i \\ &\text{and} \\ Y_m &= \sum_{i=k+2}^{n-q-1} (kn - im + m)p_i \end{aligned}\right\} \tag{9}$$

Thus (7) can be written as

$$nV = mS_1^2 + (n-m)S_2^2 + \frac{[(n-m)Q - Y_m]^2}{nm(n-m)} \tag{10}$$

As mentioned above, the first k and the last q positions are fixed. To fix the position (k+1) or (n-q), we need to know the change in variance due to interchange of jobs (k+1) and (n-q). In the following theorem we find the change in the variance due to the interchange of jobs (k+1) and (n-q) when the first k and the last q positions are fixed.

Theorem 1: Let V be the variance of completion times for the sequence R : $p_1, p_2, \ldots. p_n$. Let V^* be the variance when p_{k+1}, p_{n-q} are interchanged in sequence R. Then

$$\Delta = n(V-V^*) = \frac{p_{k+1}-p_{n-q}}{n}[2Y_m-2(n-m)T+(n-m)(k-q+1)(p_{k+1}+p_{n-q})] \tag{11}$$

where Y_m is given by (9) and

$$T = \sum_{i=n-q+1}^{n}(n-i+1)p_i - \sum_{i=1}^{k}(i-1)p_i \tag{12}$$

<u>Proof:</u> The proof is given in the appendix.

From Schrage [6], the job with the largest processing time is placed at first position i.e. $p_{(1)} = p_1$. By a result of Merten and Muller [5], there exists an optimal sequence in which $p_{(2)}$ is at the last position. Because of the V-shaped property of optimal sequence, the job with $p_{(3)}$ can be either at second position or at $(n-1)^{th}$ position. Schrage conjectured that it is at the second position. We prove below that it is true for $n \leq 18$ jobs. We could not establish the result or prove otherwise for $n > 18$.

<u>Theorem 2:</u> For $n \leq 18$ jobs, the job with third largest processing time will be at the second position.

<u>Proof:</u> The proof of the theorem 2 is given in appendix.

Using (11), we determine the optimal sequences for $n = 6$ and $n = 7$.

<u>Theorem 3:</u> For $n = 6$ jobs the optimal sequence is

$p_{(1)}, p_{(3)}, p_{(4)}, p_{(5)}, p_{(6)}, p_{(2)}$ or

$p_{(1)}, p_{(3)}, p_{(4)}, p_{(6)}, p_{(5)}, p_{(2)}$ according as

$$\sum_{i=1}^{6} p_i + p_{(3)} + 3p_{(4)} \lesseqgtr 3p_{(2)} + p_{(1)}.$$

Proof: By Schrage [6], Merten and Muller [5] and Theorem (2) we start with the sequence

$$p_{(1)},\ p_{(3)},\ \underline{\ ?\ },\ \underline{\ ?\ },\ \underline{\ ?\ },\ p_{(2)}.$$

Now using the result (11), we examine the possible interchange of p_3 and p_5. Here $k = 2$, $q = 1$, $m = 4$. We have from (11),

$$\Delta = \frac{p_3 - p_5}{6}\left[\ - 4(p_6 - p_2)\right]$$

Here $p_6 = p_{(2)}$ and $p_2 = p_{(3)}$. Therefore $p_6 - p_2 > 0$. Hence for $p_3 \leq p_5$, $\Delta \geq 0$. Thus interchanging $p_5 \geq p_3$ will always reduce the variance. Hence $p_3 = p_{(4)}$. Thus we have

$$p_{(1)},\ p_{(3)},\ p_{(4)},\ \underline{\ ?\ },\ \underline{\ ?\ },\ p_{(2)}.$$

Using the result (11) for $k = 3$, $q = 1$ and $m = 5$, we have

$$\Delta = \frac{p_4 - p_5}{6}\left[\sum_{i=1}^{6} p_i + p_{(3)} + 3p_{(4)} - 3p_{(2)} - p_{(1)}\right]$$

For $p_4 \leq p_5$, $\Delta \geq 0$ only if

$$\sum_{i=1}^{6} p_i + p_{(3)} + 3p_{(4)} \leq 3p_{(2)} + p_{(1)}$$

Thus interchange of $p_{(5)} = p_5$ is done with p_4 if the above condition is true. i.e. the optimal sequences given are obtained according as the above condition.

When there are seven jobs, we need to examine five sequences, viz.,

(1) $p_{(1)}, \; p_{(3)}, \; p_{(5)}, \; p_{(6)}, \; p_{(7)}, \; p_{(4)}, \; p_{(2)}.$

(2) $p_{(1)}, \; p_{(3)}, \; p_{(4)}, \; p_{(5)}, \; p_{(6)}, \; p_{(7)}, \; p_{(2)}.$

(3) $p_{(1)}, \; p_{(3)}, \; p_{(4)}, \; p_{(5)}, \; p_{(7)}, \; p_{(6)}, \; p_{(2)}.$

(4) $p_{(1)}, \; p_{(3)}, \; p_{(4)}, \; p_{(7)}, \; p_{(6)}, \; p_{(5)}, \; p_{(2)}.$

(5) $p_{(1)}, \; p_{(3)}, \; p_{(4)}, \; p_{(6)}, \; p_{(7)}, \; p_{(5)}, \; p_{(2)}.$

The flow chart for n = 7 jobs is given Figure 1.

Insert Figure 1

<u>Remarks:</u> It can be verified that

1. If $3(p_{(2)} - p_{(3)}) \leq \min_{\substack{\text{overall}\\ \text{possible}\\ \text{pairs}}} 2(p_4 - p_5)$

   then $p_{(4)} = p_6$.

2. If $3(p_{(2)} - p_{(3)}) \geq \max_{\substack{\text{overall}\\ \text{possible}\\ \text{pairs}}} 2(p_4 - p_5)$

   then $p_{(4)} = p_3$.

Flow-Chart for n = 7 Jobs

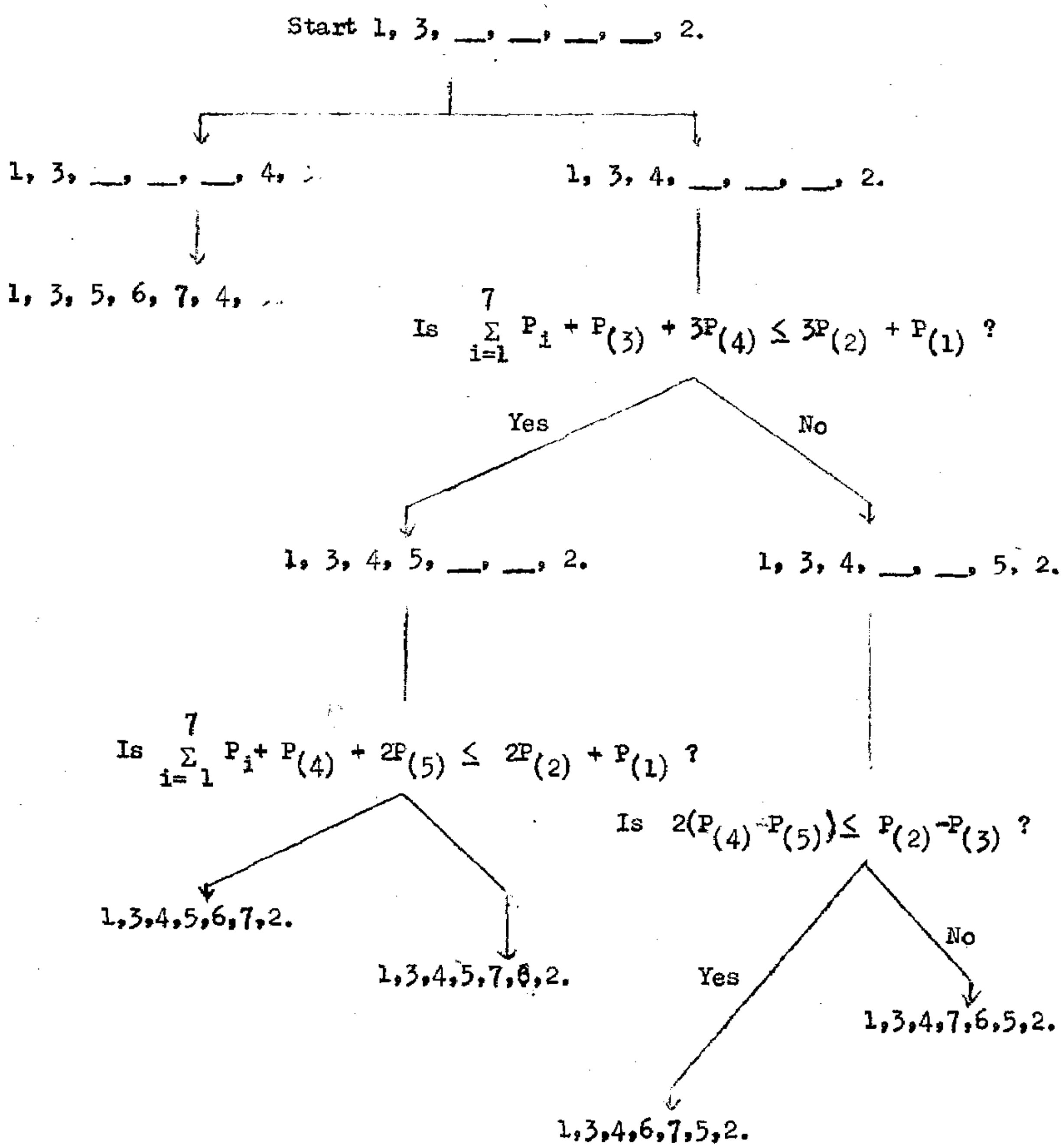


Figure .1

If one of these two conditions are true the position of $p_{(4)}$ is fixed and then we obtain an optimal sequence under conditions mentioned in Figure 1. We need not examine all five sequences.

Heuristic Procecure for General n:

In this section we develop a heuristic procedure to find the best sequence. This method checks the possibility of interchanges and accordingly positions at left or right tail of the sequence are fixed. We use the formula of partitioning variance and find the change in variance due to interchanging positions at some stage by (11). We have from (11),

$$\Delta = \frac{p_{k+1}-p_{n-q}}{n}[2Y_m-2(n-m)T+(n-m)(k-q-1)\ (p_{k+1}+p_{n-q})]$$

With Y_m and T given by (9) and (12) respectively.

We first start with the sequence with $p_1 = p_{(1)}$, $p_n = p_{(2)}$.

Rest of the positions in the sequence are arbitrary, thus for some given sequence we calculate Δ and set the positions according to the sign of Δ.

1. If $\Delta \leq 0$, then set the position (k+1) by $p_{k+1} = p_{k+1}$, for the next iteration take $k = k+1$, $q = q$, $m = m+1$ and $p_{n-q} = p_{n-q}$.

2. If $\Delta > 0$, then set the position $(n-q)$ at the right tail of the sequence i.e. $p_{n-q} = p_{k+1}$. For the next iteration take $k = k$, $q = q+1$, $m = m+1$ and $p_{k+1} = p_{n-q}$.

3. Do (1) and (2) for $m = 3, \ldots\ldots (n-1)$.

4. Repeat the steps (1), (2) and (3) till all the values of Δ at each stage of checking interchanges are ≤ 0.

All the values of $\Delta \leq 0$ indicate that no single interchange reduces the variance. For the sequences given by Kanet [3], we have found the optimal sequences by this method. Results are compared with the results obtained by SMV (Kanet's) method and with the best ones by Eilon and Chowdhary [2]. Table-1 shows the results obtained.

Remarks: To reduce the number of iterations in the proposed procedure, we can start with a good V-shaped sequence. For example, initial sequence could be,

$$p_{(1)}, p_{(3)}, p_{(5)}, \ldots\ldots\ldots , p_{(6)}, p_{(4)}, p_{(2)}.$$

PART - II

RANDOM CASE

It may however happen that the processing times of jobs are not known in advance. We therefore consider the situation when the processing times of the jobs are random variables with known distributions. Our criterion here is to schedule the n jobs so that the expected variance is minimum. We obtain some general results under certain assumptions on EP_i and EP_i^2 pertaining to the distributions of the processing times.

Let $f(P_i)$ be the probability density function of P_i, $i = 1, \ldots\ldots n$. Let

$$EP_i = \beta_i \qquad i = 1, \ldots\ldots n$$

and assume that (13)

$$EP_i^2 = g(\beta_i) = g_i, \qquad i = 1, \ldots\ldots n$$

i.e. the second raw moment of P_i is a function of the mean. We further assume that for every i, j

$$\beta_i \leq \beta_j \Longleftrightarrow g_i \leq g_j \tag{14}$$

β_i indicates the expected processing time of the job at position i.

$\beta_{(i)}$ indicates the order of job i in a sequence, when β_i s are arranged in descending order.

Thus $\beta_{(1)} \geq \beta_{(2)} \geq \ldots\ldots\ldots\ldots \geq \beta_{(n)}$.

Similar interpretation is for notations g_i and $g_{(i)}$.

From (3) we have

$$E(nV) = \sum_{i=2}^{n} \frac{(n-i+1)(i-1)}{2} EP_i^2 + \frac{2}{n} \sum_{i=2}^{n-1} (i-1)EP_i \sum_{j=i+1}^{n} (n-j+1)EP_j$$

Denoting nV by N, from (13), we have

$$E(N) = \sum_{i=2}^{n} \frac{(n-i+1)(i-1)}{n} g_i + \frac{2}{n} \sum_{i=2}^{n-1} (i-1)\beta_i \sum_{j=i+1}^{n} (n-j+1)\beta_j \quad (15)$$

If variance of P_i is given by σ_i^2 then

$g_i = \beta_i + \sigma_i^2$, $i = 1, \ldots\ldots n.$

Therefore (15) can be written as

$$E(N) = nV_F + \sum_{i=2}^{n} \frac{(n-i+1)(i-1)}{n} \sigma_i^2 \quad (16)$$

where V_F is the variance of the expected completion times, $F_i = \beta_1 + \ldots\ldots + \beta_i$, $i = 1, \ldots\ldots n.$

In (16), if $\sigma_i^2 = \sigma^2$ for $i = 1, \ldots. n$ then minimizing E(N) is equivalent to minimizing V_F which reduces the problem to the deterministic case. Thus we consider the problem here when all σ_i^2 are not equal.

Characteristics of Optimal Sequence:

We discuss the characteristics of optimal sequence which minimizes (15). We prove similar results proved in the deterministic case dealt with in Part-I.

Theorem 4: In an optimal sequence, the job with the largest expected processing time is at the first position.

Proof: In (15), g_1 and β_1 both have zero coefficients since p_1^2 and p_1 have zero coefficients in (3). Since

$\beta_{(1)} \geq \beta_i$ We have from (14), $g_{(1)} \geq g_i$ $i = 1, \ldots n.$

If the job with the expected processing time $\beta_{(1)}$ is not at the first position but at any other position, then the expected variance will increase since the coefficients are greater than zero for any other position. This completes the proof.

Corollary 1: For a sequence of two jobs, optimal schedule is

$$\beta_{(1)} \quad , \quad \beta_{(2)}.$$

Theorem 5: The expected variance for the schedule 1, 2, 3, ... (n-1), n is the same as that for the schedule 1, n, n-1, ... 3, 2.

Proof: As mentioned before, by Merten and Muller [5], if V is the variance of the schedule R : 1, 2, ... (n-1), n and V^*, the variance of the schedule, R^* : 1, n, n-1, .., 3, 2 then we have

$$nV = nV^*$$

Therefore $$EnV = EnV^*$$

i.e. $$EN = EN^*$$

where $N^* = nV^*$. This proves the theorem.

Remark: This theorem is true without condition (14).

Corollary 2: For a sequence of three jobs, both the sequences, $\beta_{(1)}, \beta_{(2)}, \beta_{(3)}$ and $\beta_{(1)}, \beta_{(3)}, \beta_{(2)}$ are optimal.

We state below two results for the case $n = 4$ and $n=5$. The proofs of these results are similar to the proofs of corresponding theorems in Schrage [6] and are therefore omitted.

Theorem 6: If $n > 3$, the schedule with minimum expected variance has

$$\beta_2 \geq \beta_3 \quad \text{and} \quad \beta_n \geq \beta_{n-1}.$$

Corollary 3: For a problem with four jobs, both the following sequences are optimal.

$$\beta_{(1)}, \beta_{(2)}, \beta_{(4)}, \beta_{(3)} \quad \text{and} \quad \beta_{(1)}, \beta_{(3)}, \beta_{(4)}, \beta_{(2)}.$$

Theorem 7: For problems with five jobs, both the sequences,

$$\beta_{(1)}, \beta_{(2)}, \beta_{(5)}, \beta_{(4)}, \beta_{(3)} \quad \text{and}$$

$$\beta_{(1)}, \beta_{(3)}, \beta_{(4)}, \beta_{(5)}, \beta_{(2)} \quad \text{are optimal.}$$

V-Shape Property:

Here we examine the V-shape of optimal sequence which minimizes expected variance of completion times, E(V). From (16) we have

$$E(N) = nV_F + \sum_{i=2}^{n} \frac{(n-i+1)(i-1)}{n} \sigma_i^2$$

$$= nV_F + V_\sigma, \quad \text{say.}$$

In the above expression, we know that the sequence which minimizes V_F is V-shaped by Eilon and Chowdhary [2]. Further, it can be verified by the examination of the coefficients $\frac{(n-i+1)(i-1)}{n}$ for $i = 2, \ldots n$ that the sequence which minimizes V_{σ} is also V-shaped. However, the order of the jobs in both the sequences is likely to be different. Thus we cannot prove that sequence which minimizes E(N) is V-shaped because the sequences which minimizes V_F and V_{σ} are V-shaped. Therefore we shall treat a sub-class of distributions for which

$$g_i = A\,\beta_i^2 + B\beta_i, \quad i = 1, \ldots n.$$

with $A \geq 0$, $B \geq 0$.

We prove V-shape of optimal sequence by imposing conditions on A and B. It is verified later that these conditions are satisfied by some wellknown distributions, e.g. (1) Exponential, (2) Gamma, (3) Uniform, (4) Erlang. Note that for subclass, we are considering, condition (14) is verified to be true, since

$$g_i - g_j = (\beta_i - \beta_j)[A(\beta_i + \beta_j) + B]$$

and $A \geq 0$, $B \geq 0$.

Theorem 8: Let $g_i = A\beta_i^2 + B\beta_i$, $i = 1, \ldots n$. $A \geq 0$, $B \geq 0$. Optimal sequence which minimizes expected variance of completion times is V-shaped, when

$$\delta_i = \frac{A(n+1)-2(A-1)i-2}{2n} \geq 0$$

$$\delta_k = \frac{(2-A)n+2(A-1)j-A}{2n} \geq 0 \qquad (17)$$

$i = 1, \ldots\ldots n-2, \qquad j = 3, \ldots\ldots n.$

Proof: Before we prove that the optimal sequence is V-shaped, we find the change in expected variance due to interchange of jobs i and $j = i+1$. when

$$g_i = A\beta_i^2 + B\beta_i \quad \text{and} \quad \sigma_i^2 = g_i - \beta_i^2$$

from (16), we have,

$$E(N) = V_F + \sum_{i=2}^{n} \frac{(n-i+1)(i-1)}{n}((A-1)\beta_i^2 + B\beta_i) \qquad (18)$$

Let V_F^* be the variance of expected completion times after interchanging of jobs i and j. Let $E(N^*)$ be value of E(N) after interchange of jobs i and j. Then the change Δ_{ij} in E(N) due to the change of jobs i and $j = i+1$ is given by

$$\Delta_{ij} = n(V_F - V_F^*) - \frac{n-2i+1}{n}[(A-1)(\beta_i^2-\beta_j^2) + B(\beta_i-\beta_j)] \qquad (19)$$

Obtaining the formula for $V_F - V_F^*$ from Eilon and Chowdhary [2], we have

$$\Delta_{ij} = 2(\beta_i-\beta_j)[\,F_i-\bar{F} - \frac{A(n+1)-2(A-1)i-2}{2n}\beta_i + \frac{(2-A)n+2(A-1)i-A}{2n}\beta_j - \frac{n-2i+1}{2n}B\,] \qquad (20)$$

Suppose the sequence is not V-shaped. Then there will be three consecutive jobs, i, j, k where

$$\beta_j > \beta_i, \beta_k.$$

We shall prove that interchange of i and j or j and k will reduce the variance by showing that if $\Delta_{ij} < 0$ then $\Delta_{jk} > 0$ and if $\Delta_{jk} < 0$ then $\Delta_{ij} > 0$, where $\Delta_{ij} = E(N) - E(N^*)$.

We have $\beta_i - \beta_j < 0$. Therefore $\Delta_{ij} < 0$ only if

$$F_i - \bar{F} - \frac{A(n+1)-2(A-1)i-2}{2n}\beta_i + \frac{(2-A)+2(A-1)i-A}{2n}\beta_j - \frac{n-2i+1}{2n}B > 0 \qquad (21)$$

By adding and subtracting

$$[1 + \frac{A(n+1)-2(A-1)j-2}{2n}]\beta_j \text{ and } \frac{(2-A)n+2(A-1)j-A}{2n}\beta_k$$

in (21) and noting that $F_j = F_i + \beta_j$, we have

$$F_j - \bar{F} - \frac{A(n+1)-2(A-1)j-2}{2n}\beta_j + \frac{(2-A)n+2(A-1)j-A}{2n}\beta_k - \frac{n-2j+1}{2n}B - [\delta_i\beta_i + \frac{A}{n}\beta_j + \delta_k\beta_k + \frac{\beta}{n}] > 0 \qquad (22)$$

where δ_i and δ_k are given by (17).

In the expression (22), the term in the square bracket is positive by (17). Then it is necessary that

$$F_j - \bar{F} - \frac{A(n+1)-2(A-1)j-2}{2n}\beta_j + \frac{(2-A)n+2(A-1)j-A}{2n}\beta_k$$

$$- \frac{n-2j+1}{2n} B > 0 \qquad (23)$$

Since $\beta_j - \beta_k > 0$, (20) and (23) will imply that

$$\Delta_{jk} > 0.$$

Thus we have proved that if $\Delta_{ij} < 0$, then $\Delta_{jk} > 0$.

Similarly it can be shown that if $\Delta_{jk} < 0$ then $\Delta_{ij} > 0$ under the condition (17). This proves the theorem.

<u>Some Particular Cases:</u>

It can be verified that δ_i and δ_k are ≥ 0 for the following distributions of P_i, $i = 1, \ldots.. n$.

1. <u>Exponential:</u>

$$f(P_i) = \frac{1}{\beta_i} e^{-\frac{1}{\beta_i} P_i} \qquad P_i \geq 0$$

$i = 1, \ldots. n.$

2. <u>Gamma:</u>

$$f(P_i) = \frac{1}{\beta_i} P^{\beta_i - 1} e^{-P_i} \qquad P_i \geq 0$$

$i = 1, \ldots.. n$

3. <u>Uniform:</u>

$$f(P_i) = \frac{1}{\alpha_i} \qquad \alpha_i \geq P_i \geq 0$$

$i = 1, \ldots.. n$

4. Erlang

$$f(P_i,\ \alpha_i) = \frac{(\alpha_i m)(\alpha_i m P_i)^{m-1}}{(m-1)!} e^{-\alpha_i m P_i} \qquad P_i \geq 0$$

$i = 1, \ldots\ldots n$

Specialization to Exponential Distribution:

When processing times follow exponential distributions, the algebra simplifies and we can obtain interchange formula corresponding to [20]. We have here from (16)

$$E(N) = nV_F + \sum_{i=2}^{n} \frac{(n-i+1)(i-1)}{n} \beta_i^2$$

With similar notations as Part I, the partitioning formula can be used to find the change in E(N) after the interchange of jobs corresponding to β_{k+1} and β_{n-q}. It can be verified that the change H is given by

$$H = \frac{\beta_{k+1}-\beta_{n-q}}{n}[2X_1-2(n-m)X_2+(n-m)(k-q-1)(\beta_{k+1}+\beta_{n-q})]$$

$$+ \frac{k(n-k)-(n-q-1)(q+1)}{n}(\beta_{k+1}^2 - \beta_{n-q}^2) \qquad (24)$$

where

$$X_1 = \sum_{i=k+2}^{n-q-1} (kn - im + m)\ \beta_i$$

$$X_2 = \sum_{i=n-q}^{n} (n-i+1)\beta_i - \sum_{i=1}^{k+1} (i-1)\beta_i$$

Using this result it can be proved that for $n \leq 24$ jobs, the job with third largest expected processing time will be at the second position when the job with the second largest expected processing time is at last position. In the case of exponential distribution, we obtain the optimal sequence for $n = 6$ and $n = 7$. We give only the results and the proofs are similar to the corresponding theorems in Part-I.

<u>Theorem 9:</u> For $n = 6$ jobs the optimal sequence is

(i) $\beta_{(1)}, \beta_{(3)}, \beta_{(4)}, \beta_{(5)}, \beta_{(6)}, \beta_{(2)}$ or

(ii) $\beta_{(1)}, \beta_{(3)}, \beta_{(4)}, \beta_{(6)}, \beta_{(5)}, \beta_{(2)}$ according as

$$\sum_{i=1}^{6} \beta_i + \beta_{(4)} \leq \beta_{(1)} + \beta_{(2)}.$$

When there are seven jobs, we need to examine five sequences viz.,

(1) $\beta_{(1)}, \beta_{(3)}, \beta_{(5)}, \beta_{(6)}, \beta_{(7)}, \beta_{(4)}, \beta_{(2)}$.

(2) $\beta_{(1)}, \beta_{(3)}, \beta_{(4)}, \beta_{(5)}, \beta_{(6)}, \beta_{(7)}, \beta_{(2)}$.

(3) $\beta_{(1)}, \beta_{(3)}, \beta_{(4)}, \beta_{(5)}, \beta_{(7)}, \beta_{(6)}, \beta_{(2)}$.

(4) $\beta_{(1)}, \beta_{(3)}, \beta_{(4)}, \beta_{(6)}, \beta_{(7)}, \beta_{(5)}, \beta_{(2)}$.

(5) $\beta_{(1)}, \beta_{(3)}, \beta_{(4)}, \beta_{(7)}, \beta_{(6)}, \beta_{(5)}, \beta_{(2)}$.

The flow chart for $n = 7$ jobs in the case of exponential distribution is given in Figure 2.

Insert Figure 2

Flow-Chart for n = 7 Jobs (Exponential Distribution)

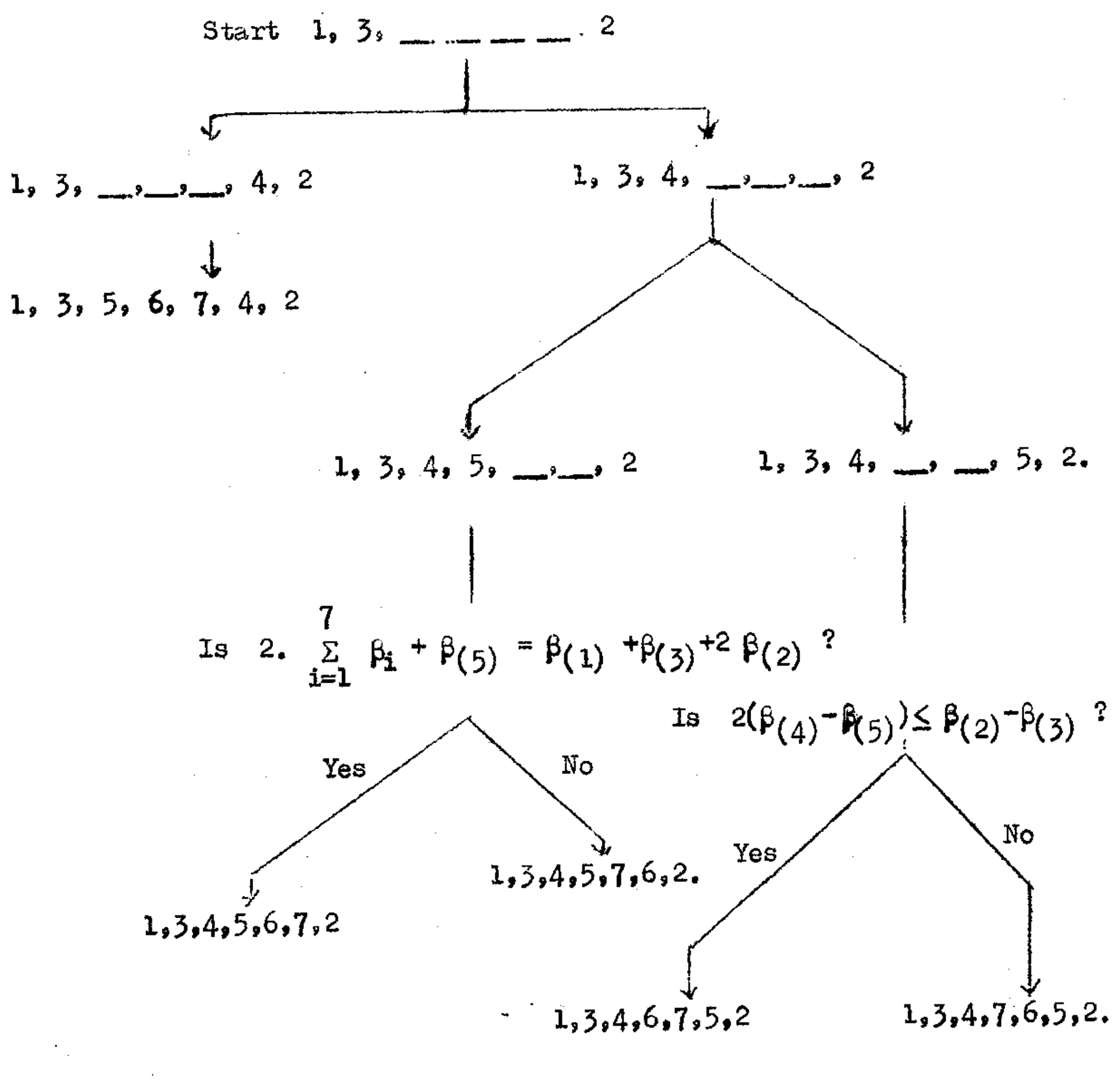


Figure 4.2

Appendix

Proof of Theorem 1

From (9) we have

$$nV = mS_1^2 + (n-m)S_2^2 + \frac{[(n-m)Q - Y_m]^2}{nm(n-m)}$$

In the above expression, if p_{k+1} and p_{n-q} are interchanged, then the change in nV is only due to the factor Q. The remaining terms will remain the same after changing the jobs. Let Q^* be the value of Q after interchanging p_{k+1} and p_{n-q}. Then we have

$$Q^* = \sum_{i=n-q+1}^{n} (n-i+1)p_i - \sum_{i=1}^{k} (i-1)p_i + (q+1)p_{k+1} - k\, p_{n-q}$$

Let nV^* be the value of nV after interchanging the jobs then the difference in variance is given by

$$n(V-V^*) = \frac{[(n-m)\ Q - Y_m]^2}{nm\ (n-m)} - \frac{[(n-m)Q^* - Y_m]^2}{nm(n-m)}$$

$$= -\frac{(n-m)\ (Q - Q^*)\ [(n-m)(Q + Q^*) - 2Y_m]}{nm(n-m)}$$

$$= \frac{p_{k+1} - p_{n-q}}{n}\ [2Y_m - 2(n-m)T + (n-m)(k-q-1)(p_{k+1}+p_{n-q})$$

Appendix

Proof of Theorem 2

As discussed before considering the sequence $p_{(1)}$, ? ,......... ?, $p_{(2)}$, we have $k = 1$, $q = 1$ and $m = 3$. $p_{(3)}$ can be at either second or (n-1) position. We examine the interchange between p_2 and p_{n-1}. From (10) we have

$$\Delta = \frac{p_2-p_{n-1}}{n}\left[2\sum_{i=3}^{n-2}(n-3i+3)p_i - (n-3)(2p_n + p_{n-1} + p_2)\right]$$

If $p_2 \leq p_{n-1}$, $\Delta \geq 0$ only if

$$Z = 2\sum_{i=3}^{n-2}(n-3i+3)p_i-(n-3)(2p_n+p_{n-1} + p_2) \leq 0.$$

If $p_{n-1} = p_{(3)}$, theninterchanging p_{n-1} with $p_2 \leq p_{(3)}$ does not increase the variance if $Z \leq 0$. We examine the values of n for which $Z \leq 0$, when $p_{n-1} = p_{(3)}$. Thus the problem is reduced to show that $Z \leq 0$. This result is obtained by comparing the coefficients of p_i s, $i = 3, \ldots\ldots n$ in Z. We have

$$Z = 2.[(n-6)p_3+(n-q)p_4 + \ldots -(2n-q)p_{n-2}]-(n-3)(p_2+2p_n+p_{n-1}) \quad \text{(a)}$$

In the above expression, for the index

$$i \leq I(\tfrac{n}{3} + 1) = u,$$

the coefficient of p_i is non-negative, where I(x) denotes the largest integer $\leq x$. Now in the expression (a) consider the terms with negative coefficients as

$$-2\ (2n-9)p_{n-2} -(n-3)(p_2 + p_{n-1} + 2p_n).$$

We have $p_n = p_{(2)}$, $p_{n-1} = p_{(3)}$. Then $p_{(4)} = p_2$ or $p_{(4)} = p_{n-2}$. We examine both the possibilities separately.

(1) $p_{(4)} = p_2$

In this case,

$$(n-3)(p_2+p_{n-1}+2p_n)=(n-3)(p_{(4)}+p_{(3)}+2p_{(2)}) \geq 4(n-3)p_{(4)} \qquad (b)$$

The contribution of positive terms in Z is

$$2.\sum_{i=3}^{u}(n-3i+3)p_i \leq 2p_{(4)}.\sum_{i=3}^{u}(n-3i+3) \qquad (c)$$

We obtain $S = \sum_{i=3}^{u}(n-3i+3) = 2n(u-2)-3u^2+3u+6.$

From (b) and (c) we have

$$Z \leq [2n(u-2)-3u^2+3u+6-4(n-3)]p_{(4)}.$$

The right side of the above expression is ≤ 0 if $2n(u-2)-3u^2+3u+6 \leq 4(n-3)$. It can be verified that this is true for $n \leq 18$.

2. $p_{(4)} = p_{n-2}$.

Here we have

$$(n-3)(p_{n-1}+2p_n)+2(2n-9)p_{n-2} = (n-3)(p_{(3)}+2p_{(2)})+2(2n-9)p_{(4)},$$

and

$$(n-3)(p_{(3)}+2p_{(2)})+(4n-18)p_{(4)} \geq (7n-27)p_{(4)}. \qquad (d)$$

From (b) and (d)

$$Z \leq [2n(u-2)-3u^2+3u+6-7n+27]\ p_{(4)}.$$

The right side of the above expression is ≤ 0 if

$$2n(u-2)-3u^2+3u+6-7n+27 \leq 0$$

i.e. $\quad n(2u-11) \leq 3(u^2-u-11)$

This is true for $n \leq 26$.

Thus from both the cases (1) and (2) we consider that for $n \leq 18$, the theorem 2 is true.

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