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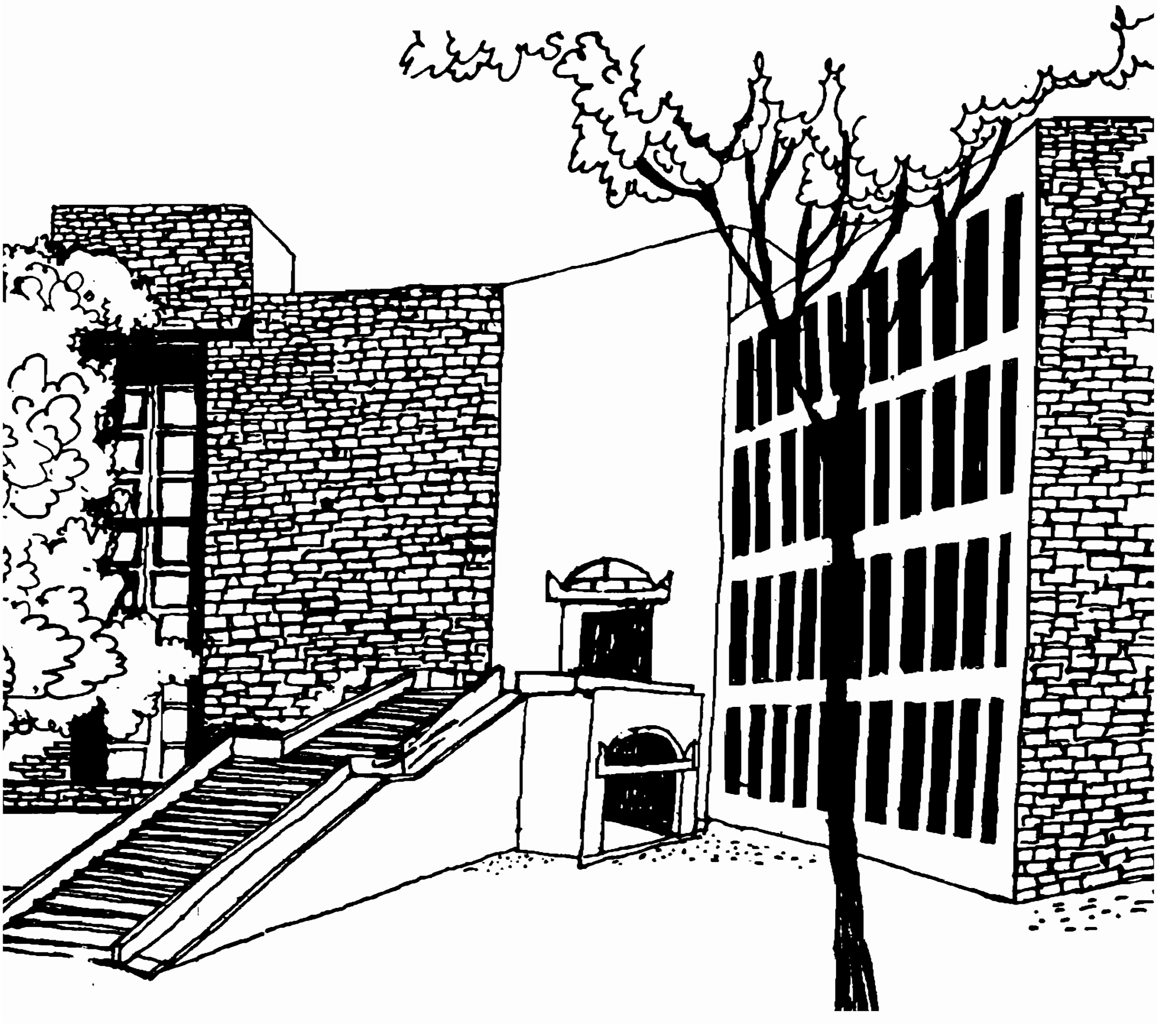

A RECONSIDERATION OF SOME PROPERTIES OF SOLUTIONS FOR TWO DIMENSIONAL CHOICE PROBLEMS

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Abstract

In this paper, we take up the outstanding problem of axiomatically characterizing what we have referred to in the paper as the additive choice function on the classical domain for choice problems. Apart from an impossibility result for the additive choice function, there is an axiomatic characterization, which as a by-product provides a counter example to a conjecture for the egalitarian choice function. In an appendix, we provide a proof of an axiomatic characterization of the egalitarian choice function using a superadditivity axiom. Further we show several non-rationalizability properties of utilitarian consistent solutions.

In this paper, we also provide proofs of axiomatic characterizations of the family of non-symmetric Nash choice functions and the family of weighted hierarchies of choice functions. Our conclusion is that earlier axiomatizations are essentially preserved on the classical domain for choice problems. The proofs are significant in being non-trivial and very dissimilar to existing proofs on other domains.

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The results on the additive choice function and Appendix A, were entirely inspired by a very fruitful discussion that I had with Hans Peters immediately after the presentation. However, none of those mentioned above should be held responsible for whatever errors that possibly still do remain.

A RECONSIDERATION OF SOME SOLUTIONS FOR TWO DIMENSIONAL CHOICE PROBLEMS

1 Introduction

Choice theory which dawned with the seminal paper of Nash written in 1950, has by now developed into a well defined body of mathematics, concerned with choosing a point from a compact, convex, comprehensive feasible subset of the non-negative orthant of a finite dimensional Euclidean space, each such feasible set admitting a strictly positive vector. Axiomatic choice theory is concerned with the axiomatic characterization of rules which assign an alternative to each such choice problem in a given family of choice problems. We shall here be concerned with two dimensional choice problems.

Following the choice function suggested by Nash, the other well known choice functions are the relative egalitarian due to Kalai and Smorodinsky [1975], egalitarian due to Kalai [1977], lexicographic egalitarian due to Chun and Peters [1988], equal loss due to Chun [1988], lexicographic equal loss due to Chun and Peters [1991] and the equal area due to Anbarci and Bigelow [1994]. Some of the other choice functions have been studied on more relevant domains in Lahiri [1996]. However, the simplest of all solutions i.e., the one which maximizes the sum of the coordinates from amongst all feasible vectors has been a rather mute spectator of a spectacular pageantry in which all these other choice functions participate. Except for a significant axiomatic characterization by Myerson [1981], very little attention has been devoted to this choice function: the utilitarian choice function. The reason is that this choice function (as a single valued mapping) is not well defined for a very large class of meaningful and non pathological choice problems. The purpose of this paper is to suggest a way out of this difficulty, so that much of applied research which uses maximization of the sum of the coordinates of vectors in a feasible set of vectors will now have a theoretical underpinning. However, it is observed in the paper, that one can easily prove several results showing that utilitarian consistent solutions are in general not rationalizable by continuous social welfare orderings or social welfare functions. We also suggest a variant of a choice function due to Cao [1981], which is also well defined on the larger domain and yet satisfies scale translation covariance property. Some remarks about related results due to Peters [1986a] are given, to put earlier results in proper perspective. In an appendix to this paper we prove a variant of a result in Peters [1986a], which is valid on our domain.

The family of non-symmetric Nash choice functions, which was proposed for the first time in the seminal work of Harsanyi and Selten [1972], has been axiomatically characterized in almost the same way that Nash himself characterized its symmetric ancestor in his by now historic 1950 paper. A more recent and thorough investigation of the family of choice functions characterized by a weighted hierarchy (and containing the family of non-symmetric Nash choice functions) is the work of Peters [1986b]. There an additional axiom called the consistency axiom is used, which however is not required for two dimensional choice problems. All the above mentioned characterizations of the non-symmetric family under discussion, rely heavily on an assumption which has often been questioned from various quarters: Nash's Independence of Irrelevant Alternatives Assumption (NIIA).

There has been several attempts to free the characterization of the Nash choice function from the grip of NIIA. Of interest in the present paper is a characterization for two dimensional choice problems presented in Thomson [1981], where instead of NIIA an assumption called Independence of Irrelevant Expansions (IEE) has been used. Interest in choice theory had since then shifted largely to the multidimensional cases and even more to choice problems with varying dimensions. A recent revival of interest in the two dimensional case (and solely that) is seen in the paper by

Bossert [1994], where once again NIIA is used to characterize rational choice functions. Our theorem 3 in the present paper is an easy and valid extension of Thomson's original result to the non-symmetric cases.

In Peters [1986b] can be found a characterization of a family of choice functions determined by a weighted hierarchy for two dimensional choice problems using a slightly weakened version of Thomson's Independence of Irrelevant Expansions assumption. However, the domain chosen for the result deviates considerably from the conventional domain used by Thomson [1981] or Bossert (1994), in that it assumes that every choice problem admits infinite free disposability. Now, this is an assumption whose worth or meaningfulness depends on the context. If we assume that each choice problem represents a multisectoral investment planning problem for instance (i.e., dividing a dollar between several sectors, the returns being measured by concave, non-decreasing, non-constant and continuous revenue functions), then the kind of domain assumed in Peters [1986b] for the present purpose is not quite meaningful. That the set of investment planning problems is isomorphic to the domain of choice problems assumed in this paper, is however a result established in Lahiri (1994). So, the natural question that crops up is whether the result established by Peters is valid when the domain (as in the present paper) consists of non-empty, compact, convex, comprehensive subsets of two dimensional Euclidean spaces, each such set admitting a strictly positive vector. A cursory look at the proof of the result in Peters [1986a], shows that it is very dependent on his choice of domain. In fact, a couple of lemmas simply do not have any meaning in our framework. What is however noteworthy, is our Theorem 4: the original result continues to hold. The choice functions determined by weighted hierarchies, are the only choice functions which satisfy the assumptions suggested by Peters.

2 The Model

We consider two dimensional choice problems only. A (two dimensional) choice problem is a non-empty subset S of $\mathbb{R}^2_+$ (: the non-negative quadrant of two dimensional Euclidean space), satisfying the following properties:

i) S is compact (: closed and bounded), convex

ii) S is comprehensive i.e. $0 \leq y \leq x \epsilon S \rightarrow y \epsilon S$

iii) there exists $x \epsilon S$ such that $x > 0$ (i.e. if $x = (x_1, x_2)$ then $x_1 > 0$, $x_2 > 0$). Let Σ^2 be the class of all choice problems.

A choice function (or solution) is a function $F : \Sigma^2 \rightarrow \mathbb{R}^2_+$ such that $F(S) \epsilon S \; \forall \; S \epsilon \Sigma^2$.

Given $S \epsilon \Sigma^2$, let $u(S) \equiv \{x \epsilon S / x_1 + x_2 \geq y_1 + y_2 \; \forall \; y = (y_1, y_2) \epsilon S\}$. u(S) is non-empty for all $S \epsilon \Sigma^2$. Further u(S) is a compact convex subset of $\Delta_c \equiv \{x \epsilon \mathbb{R}^2_+ / x = (x_1, x_2), \; x_1 + x_2 = c\} \; \forall \; S \epsilon \Sigma^2$ for some c>0. However, u(S) is in general not a singleton.

<u>Example</u>: Let $S = \{x \epsilon \mathbb{R}^2_+ / x = (x_1, x_2), \; x_1 + x_2 \leq 1\}$. Then $u(S) = \Delta_1$.

Let $a_1(S) \equiv \max \{x_1 / \exists \; x_2 \geq 0 \; with \; (x_1, x_2) \epsilon u(S)\}$,

$b_1(S) \equiv \min \{x_1 / \exists \; x_2 \geq 0 \; with \; (x_1, x_2) \epsilon u(S)\}$.

Let $a(S) = (a_1(S), a_2(S))$,

$b(S) = (b_1(S), b_2(S)) \epsilon U(S)$

Clearly, a(S) and b(S) are well defined for all $S \epsilon \Sigma^2$ and $u(S) = \{ta(S) + (1-t)\, b(S) / t \epsilon [0,1]\}$.

We define the <u>additive choice function</u> $\bar{A} : \Sigma^2 \to \mathbf{R}^2_+$ as follows:

$\bar{A}(S) = \frac{1}{2}(a(S) + b(S))\ \forall\ S \epsilon \Sigma^2$.

We are basically interested in the axiomatic characterization of this choice function, which is nothing but the expected value of the random vector which has a uniform distribution on u(S).

3 Some Axioms

Let $F : \Sigma^2 \to \mathbf{R}^2_+$ be a choice function.

1) <u>Weak Pareto Optimality (WPO):</u>
$\forall\ S \epsilon \Sigma^2,\ F(S)\ \epsilon\ W(S)$, where $W(S) \equiv \{x \epsilon S / y > x \to y \notin S\} \forall\ S \epsilon \Sigma^2$.

2) <u>Pareto Optimality (PO):</u>
$\forall\ S \epsilon \Sigma^2,\ F(S)\ \epsilon\ P(S)$, where
$P(S) \equiv \{x \epsilon S / y \geq x,\ y \epsilon S \to y = x\} \forall\ S \epsilon \Sigma^2$.

3) <u>Scale Translation Covariance (STC):</u>
$\forall\ S \epsilon \Sigma^2,\ \forall\ c \epsilon \mathbf{R}^2_{++}$ *if* $c = (c_1, c_2)$ then
$F(cS) = (c_1 F_1(S),\ c_2 F_2(S))$, given that
$cS = \{(c_1 x_1,\ c_2, x_2) / (x_1, x_2) \epsilon S\}$.

4) <u>Homogeneity (HOM):</u>
$\forall\ S \epsilon \Sigma^2,\ \forall\ t > 0,\ F(tS) = tF(S)$, where
$tx = (tx_1, tx_2)\ \forall\ x = (x_1, x_2) \epsilon R^2_+$ and $tS = \{tx / x\ \epsilon\ S\}$.

5) <u>Additivity (Addi):</u>
$\forall\ S \epsilon \Sigma^2,\ T \epsilon \Sigma^2,\ F(S + T) = F(S) + F(T)$.

6) Super Additivity (S Addi):

$\forall\, S, T \in \Sigma^2,\ F(S+T) \geq F(S) + F(T)$.

7) Partial Super Additivity (PS Addi):

$\forall\, S, T \in \Sigma^2,\ F(S+T) \geq F(S)$.

8) Nash's Independence of Irrelevant Alternatives (NIIA):

$\forall\, S, T \in \Sigma^2,\ S \subset T,\ F(T) \in S \rightarrow F(S) = F(T)$.

9) Translation Covariance (TC):

$\forall\, S \in \Sigma^2,\ c \in \mathbf{R}^2$ if $S(c) = \{y \in \mathbf{R}^2 / y \leq x + c,\ x \in S\}$, then $F(S(c)) = F(S) + c$.

10) Symmetry (SYM):

$\forall\, S \in \Sigma^2$ such that $(x_1, x_2) \in S \leftrightarrow (x_2, x_1) \in S,\ F_1(S) = F_2(S)$.

11) Convex Linearity (C. Lin):

$\forall\, S, T \in \Sigma^2,\ F(\alpha S + (1-\alpha) T) = \alpha F(S) + (1-\alpha) F(T)$ if $\alpha \in [0, 1]$.

12) Binary Additivity (B. Addi):

$\forall\, S, T \in \Sigma^2$ with $U(S) = \{\bar{A}(S)\}$ and $U(T) = \{\bar{A}(T)\}$ if

V = comprehensive convex hull {S, T}, then

$F(V) = \frac{1}{2}[F(S) + F(T)]$ if $F_1(S) + F_2(S) = F_1(T) + F_2(T)$.

Let us first mention that $\bar{A}$ does not satisfy STC and NIIA.

Example:

Let $T = \{x \in \mathbf{R}^2 / (x_1, x_2) = x,\ x_1 + x_2 \leq 1\}$,

$S = \text{Convex hull}\left\{(0,0),\ (0,1),\ \left(\frac{1}{2}, \frac{1}{2}\right), \left(\frac{1}{2}, 0\right)\right\}$.

Clearly $S \subset T$ and $\bar{A}(T) = \left(\frac{1}{2}, \frac{1}{2}\right) \in S$. However $\bar{A}(S) = \left(\frac{1}{4}, \frac{3}{4}\right)$. Thus $\bar{A}$ does not satisfy NIIA.

We will however, modify $\bar{A}$ somewhat later to take care of STC. Observe that:

i) PO → WPO
ii) STC → HOM
iii) Addi → S Addi → PS Addi
iv) Addi + HOM → C. LIN

4 A Result on the Additive Choice Function

Theorem 1:

The only choice function on Σ^2 to satisfy PO, SYM, C.LIN and B. Addi is $\bar{A}$.

Proof:

The proof that if F satisfies PO, SYM and C.LIN, then $F(S) \in \underset{x \in S}{argmax}\left[x_1 + x_2\right] \; \forall S \in \Sigma^2$ is the relevant portion of the proof of theorem 1 in Myerson [1981]. If in addition F satisfies B.Addi the following argument holds:

Let $V \in \Sigma^2$ and let $h_i(V) = \max\{x_i / x \in V\}$, $i = 1, 2$. Suppose $\{\bar{A}(V)\}$ is a strict subset of U(V). (If $u(V) = \{\bar{A}(V)\}$, there is nothing more to be proved).

Case 1: $a(V) \in \mathbb{R}^2_+ \setminus \mathbb{R}^2_{++}$, $b(V) \in \mathbb{R}^2_+ \setminus \mathbb{R}^2_{++}$.

In this case $V = \Delta_c$ for some $C > 0$. By WPO and SYM, $F(V) = \bar{A}(V)$

Case 2: $a(V) \in \mathbf{R}^2_{++}$, $b(V) \in \mathbf{R}^2_{++}$

Let S = Convex comprehensive hull $\{(0, h_2(V)), \{x \in V / x_2 \leq a_2(V)\}\}$.

T = Convex comprehensive hull $\{(h_1(V), 0), \{x \in V / x_1 \leq b_1(V)\}\}$.

Clearly V = Convex comprehensive hull {S,T}

Further, $u(S) = \{\bar{A}(S)\} = \{a(V)\}$, $u(T) = \{\bar{A}(T)\} = \{b(V)\}$.

Thus F(S) = a(V), F(T) = b(V).

By *B.Addi*, $F(V) = \bar{A}(V)$.

Case 3: $a(V) \in \mathbf{R}^2_{+} \setminus \mathbf{R}^2_{++}$, $b(V) \in \mathbf{R}^2_{++}$

In this case let T be as in Case 2 and let

$S = \{x \in V / x_2 \leq a_2(V)\}$

Once again V = Comprehensive convex hull {S,T} and from here on the argument is as in Case 2.

Case 4: $a(V) \in \mathbf{R}^2_{++}$, $b(V) \in \mathbf{R}^2_{+} \setminus \mathbf{R}^2_{++}$

In this case let T be as in Case 2 and let $S = \{x \in V / x_2 \leq (V)\}$

Again V = Comprehensive convex hull {S,T} and the resulting argument is as in Case 2.

Thus $F(V) = \bar{A}(V)$ in all cases Q.E.D.

Remarks:

1) The theorem due to Myerson [1981] which we refer to in our proof is valid only on a subdomain of Σ^2 for which $u(S) = \{\bar{A}(S)\}$. However, the same proof works for us.

2) We have shown that $\bar{A}$ satisfies PO, SYM, HOM and Addi. Thus $\bar{A}$ satisfies PO, SYM, HOM, and PS. Addi. Peters [1986] contains a theorem to the effect that the egalitarian solution due to Kalai [1977], is the only solution to satisfy WPO, SYM, HOM and PS. Addi. However, his domain is a nonconventional one and is different from ours. On our domain the egalitarian solution satisfies WPO, SYM, HOM and PS. Addi. as well. Thus a uniqueness result using WPO, SYM, HOM and PS. Addi on Σ^2 is clearly not available. It is interesting to note that our domain Σ^2 is naturally implied by the interesting discussion on Axiomatic Bargaining contained in Moulin [1983]. Moulin [1983], considers a domain which is a strict subset of Σ^2. However, all choice problems in Σ^2 can be obtained as the limit in the Hansdorff topology of a sequence of increasing choice problems considered by Moulin [1983].

3) Since $\bar{A}$ does not satisfy NIIA, the interesting axiomatic characterization on the subdomain of Σ^2 defined by $\left\{S \epsilon \Sigma^2 / u(S) = \{\bar{A}(S)\}\right\}$ using PO, SYM, TC and NIIA which is there in Exercise 3.9 of Moulin [1983] fails to generalize.

Proposition 1:

On Σ^2 there exists no choice function which satisfies WPO, SYM, TC and NIIA.

<u>Proof:</u>

Let $a = \left(\frac{1}{4}, \frac{3}{4}\right)$, $b = \left(\frac{1}{2}, \frac{1}{2}\right)$ and S = comprehensive convex hull of $\{a,b\}$.

Now $S \subseteq \Delta_1$

Suppose towards or contradiction that there exists a choice function F which satisfies the above assumptions. Then by WPO, S and SYM, $F(\Delta) = \left(\frac{1}{2}, \frac{1}{2}\right)$ and by NIIA, $F(S) = \left(\frac{1}{2}, \frac{1}{2}\right) = b$.

Now let $C = \left(\frac{3}{4}, \frac{1}{4}\right)$ and T = comprehensive convex hull of $\{a + c, . b + c\}$.

Then T = S(c) as defined in the Translation Covariance assume.

Now $T \subset \Delta_2$ and $F(\Delta_2) = (1,1) = a + c$ by WPO and SYM. By NIIA, F(T) = (1,1) = a + c.

By TC, $F(T) = F(S) + c = b + c = \left(1\frac{1}{4}, \frac{3}{4}\right) \neq (1,1)$.

This consideration establishes the desired nonexistence. Q.E.D.

We define the following choice function $A^* : \Sigma^2 \to \mathbf{R}^2_+$ which satisfies both NIIA and SYM:

Let $\Gamma = \{ (x_1, x_2) \in \mathbf{R}^2_+ / x_1 = x_2 \}$.

Given $S \in \Sigma^2$, let $A^*(S) = \Gamma \cap u(S)$ if $\Gamma \cap u(S) \neq \phi$

$= b(S)$ if $x_1 > x_2 \; \forall (x_1, x_2) \in u(S)$

$= a(S)$ if $x_1 < x_2 \; \forall (x_1, x_2) \in u(S)$

It is easy to see that $A^{*} : \Sigma^{2} \to \mathbf{R}^{2}_{+}$ satisfies Pareto Optimality.

The subsequent results are related to the results reported in Peters and Wakker (1991), Bossert (1994) and Bossert (1996).

Let R be a binary relation on $\mathbf{R}^{2}_{+}$ which is reflexive (i.e. x R x $\forall x \in \mathbf{R}^{2}_{+}$), transitive

$(i.e.\ xRy \wedge yRz \to xRz\ \forall\ x,y,z \in \mathbf{R}^{2}_{+})$ and total

$(i.e.\ x,y \in \mathbf{R}^{2}_{+},\ x \neq y \to xRy \vee yRx)$. Such an R is called an ordering.

An ordering R is said to be continuous, if

$\forall\ x \in \mathbf{R}^{2}_{+},\ \{ y \in \mathbf{R}^{2}_{+} / yRx \}$ and $\{ y \in \mathbf{R}^{2}_{+} / xRy \}$, are closed.

Let $F: \Sigma^{2} \to \mathbf{R}^{2}_{+}$ F is said to be rationalizable by a continuous ordering R if

$\forall\ S \in \Sigma^{2},\ \{ F(S) \} = \{ x \in S / xRy\ \forall\ y \in S \}$.

A choice function $F: \Sigma^{2} \to \mathbf{R}^{2}_{+}$ is said to be utilitarian consistent, if $F(S) \in u(S)\ \forall\ S \in \Sigma^{2}$.

Theorem A:- Let $F: \Sigma^2 \to \mathbf{R}_+^2$ be any utilitarian consistent solution.

Then F is not rationalizable by any continuous ordering.

Proof:- Let $F: \Sigma^2 \to \mathbf{R}_+^2$ be any utilitarian consistent solution.

Towards a contradiction assume that F is rationalizable by a continuous ordering R.

Let P be the asymmetric part of R.
Let S = cch { (d,0), (0,d) }, d > 0.

Suppose F(S) >> 0. Let $\bar{x} = F(S)$ *with* $\bar{x}_1 > 0$, $\bar{x}_2 > 0$.

$\therefore \bar{x}\ P\ (d,0)$ *and* $\bar{x}\ P\ (0,d)$.

By continuity of R, y P (d, 0) and y P (0,d) $\forall$ y in a sufficiently small neighborhood N of $\bar{x}$.

Choose $\epsilon>0$ so small that, if T = cch {(d-ϵ, 0), (0, d)}, then $T \cap N \neq \phi$. Since F(T)ϵu(T),F(T)=(0,d). But,

$T \cap N \neq \phi \to y \in T$ such that y P (0, d) which is a ontradiction.

Hence, F(S) = (0, d) or (d, 0).

Suppose without loss of generality F (S) = (d, 0).
Then (d, 0) P (0, d).
By continuity of R, (d - ϵ, 0) P (0, d) for ϵ > 0 sufficiently small.

Since, if T = cch $\{(d - \epsilon, 0), (0, d)\}$, F (T) ϵ u (T) implies F (T) = (0, d), we get that (d - ϵ, 0) P F (T) which is a contradiction.

Thus F is not rationalizable by any continuous ordering.

Q.E.D.

Note:- In the above proof cch refers to comprehensive, convex hull.

Remark 4:- Thus A* is not rationalizable by any continuous ordering. Contrast this with the result in Peters and Wakker (1991), which says that if a choice function satisfies PO, CONT and NIIA, it is rationalizable by an upper semicontinuous ordering.

Remark 5:- The above theorem is easily seen to be valid for utilitarian compatible choice functions defined on the space of n-dimensional choice problems i.e. collection of nonempty, compact, convex, comprehensive subsets of $\mathbf{R}^n_+$ (: the non-negative orthant of n-dimensional Euclidean space) each of which admits a strictly positive vector.

Remark 6:- In view of our proof of the main theorem and Remark 1, the following observation is easily seen to be valid: Let $h: \mathbf{R}^n_{++} \times \mathbf{R}_{++} \to \mathbf{R}^n_+$ be a function which is homogeneous of degree one and such that $\forall (p, w) \in \mathbf{R}^n_{++} \times \mathbf{R}_{++}$, $\sum_{i=1}^{n} p_i\, h_i\,(p, w) \leq w$. Further suppose that $\forall\,(p, w) \in \mathbf{R}^n_{++} \times \mathbf{R}_{++}$ and $\forall\, x \in \mathbf{R}^n_+$ with

$\sum_{i=1}^{n} p_i x_i \leq w$, $\sum_{i=1}^{n} h_i(p,w) \geq \sum_{i=1}^{n} x_i$. Then there does not exist any total, reflexive, transitive and continuous binary relation R on $\mathbf{R}_+^n$ such that $\forall (p,w) \in \mathbf{R}_{++}^n x \mathbf{R}_{++}, \{h(p,w)\} = \{ x \in \mathbf{R}_+^n / \sum_{i=1}^{n} p_i x_i \leq w$ and $xRy \ \forall \ y \in \mathbf{R}_+^n$ with $\sum_{i=1}^{n} p_i y_i \leq w\}$.

<u>Remark 7</u>:- Let $\mathbf{R}_{++}^n = \{x \in \mathbf{R}_+^n / x_i > 0 \ \forall \ i=1,\ldots,n\}$. In view of Remark 1, a slight modification of our proof of the main theorem yields the result that if F is a choice function for n-dimensional choice problems which is weighted utilitarian consistent, then there does not exist any total, reflexive, transitive and continuous binary relation R on $\mathbf{R}_+^n$, which rationalizes F. Here, F is weighted utilitarian consistent with weights $w \in \mathbf{R}_{++}^n$ if $F(S) \in u^w(s) \equiv \{x \in S / \sum_{i=1}^{n} w_i x_i \geq \sum_{i=1}^{n} w_i y_i \ \forall \ y=(y_i,\ldots,y_n) \in S\}$ for all n-dimensional choice problems.

Infact, we can prove a slightly stronger theorem than the one proved above.

Given a choice function $F : \Sigma^2 \to \mathbf{R}^2_+$, say that it is rationalized by a social welfare function $V : \mathbf{R}^2_+ \to \mathbf{R}$ *if* $\forall S \in \Sigma^2$, $\{F(S)\} = \{x \in S / V(x) \geq V(y) \ \forall y \in S\}$.

Note: No assumptions are being made with regard to the continuity of V.

Theorem B:- If $F : \Sigma^2 \to \mathbf{R}^2_+$ is utilitarian consistent and symmetric then it cannot be rationalized by any social welfare function $V : \mathbf{R}^2_+ \to \mathbf{R}$.

Proof:- Towards a contradiction assume that $F : \Sigma^2 \to \mathbf{R}^2_+$ is utilitarian consistent and symmetric which is also rationalized by a social welfare function $V : \mathbf{R}^2_+ \to \mathbf{R}$. Let $0 < d' < d$.

Let $T_d = cch\{(0,d),\ (d,0)\}$; then $F(T_d) = (\frac{d}{2}, \frac{d}{2})$.

$T_{d'} = cch\{(0,d'),\ (d',0)\}$ implies $F(T_{d'}) = (\frac{d'}{2}, \frac{d'}{2})$.

Both the above follow from symmetry of F. Thus

$$V\left(\frac{d}{2}, \frac{d}{2}\right) > \max\{V(d,0),\ V(0,d)\}.$$

Let $T = cch\left\{(0,d),\ \left(\frac{dd'}{2d-d'},\ 0\right)\right\}$

$F(T) = (0,d)$ implies

$$V(0,d) > V\left(\frac{d'}{2}, \frac{d'}{2}\right) \text{ since } \left(\frac{d'}{2}, \frac{d'}{2}\right) \in T$$

Let $T' = cch\left\{(d,0),\ \left(0, \frac{dd'}{2d-d'}\right)\right\}$.

We get in a similar fashion.

$$V(d,0) > V\left(\frac{d'}{2},\frac{d'}{2}\right).$$

Thus $V\left(\frac{d}{2},\frac{d}{2}\right) > \max\{V(d,0),\ V(0,d)\}.$

$$\geq \min\{V(d,0),\ V(0,d)\}.$$

$$> V\left(\frac{d}{2},\frac{d'}{2}\right) > \max\{V(d',0),\ V(0,d')\}.$$

Let r(d) be a rational number between $V\left(\frac{d}{2},\frac{d}{2}\right)$ and

max $\{V(d,0),\ V(0,d)\}$, $d > 0$.

Thus r is a function from $\mathbf{R}_{++}$ to the rationals which is strictly increasing and hence one-to-one. But this is impossible. Hence the theorem.

<u>Q.E.D.</u>

The implications of this theorem, are rather powerful as we shall soon observe.

Remark 8:- The solution A* is symmetric and utilitarian consistent. Hence, by Theorem 2, it is not rationalizable, by any social welfare function.

It is easy to see that A* does not satisfy CONT.

In Peters and Wakker (1991), we have a result of fundamental importance: A sufficient condition for $F : \Sigma^2 \rightarrow \mathbb{R}^2_+$ to be rationalizable by a social welfare function is that F satisfies PO, NIIA and CONT. Thus we may conclude that there exists no utilitarian consistent solution which satisfies NIIA, SYM and CONT.

5 The Weighted Additive Choice Function

The conventional method of extending the additive choice function is to consider maximizers of the weighted sum of the coordinates. We however restrict ourselves to a particular kind of weighting system, so that the resulting solution satisfies STC.

Given $S \in \Sigma^2$, let $h(S) \equiv (h_1(S), h_2(S))$ where $h_i(S) = \max\{x_i / x \in S\}$, $i = 1, 2$. Clearly $h_i(S) > 0$, $i = 1, 2$, $\forall S \in \Sigma^2$.

Let $f_i(S) = \dfrac{1}{h_i(S)}$, $i = 1, 2$. Thus $h(f(S)S) = (1,1)$ $\forall S \in \Sigma^2$.

Let $\bar{B} : \Sigma^2 \rightarrow \mathbb{R}^2$ be a choice function defined as follows:

$$\bar{B}(S) \quad \left(h_1(S)\left[\frac{a_1(f(S)S) + b_1(f(S)S)}{2}\right],\ h_2(S)\left[\frac{a_2(f(S)S) + b_2(f(S)S)}{2}\right]\right),$$

where a_i, b_i, $i=1, 2$ are functions from Σ^2 to $\mathbb{R}$ defined earlier. Clearly $\bar{B}(S)$ corresponds to the expected value of the random vector which has a uniform distribution on the set

$$\left\{x \in S / \frac{x_1}{h_1(S)} + \frac{x_2}{h_2(S)} \geq \frac{y_1}{h_1(S)} + \frac{y_2}{h_2(S)}\ \forall (y_1, y_2) = y \in S\right\}.$$

The particular case when this set reduces to $\{\bar{B}(S)\}$ is known as the choice function due to Cao [1981].

We now invoke the following axioms:

13) Restricted Convex Linearity (RC. LIN):

$\forall S, T \in \Sigma^2$ with $h(S) = h(T)$, $F(\alpha S + (1-\alpha) T)$
$= \alpha F(S) + (1-\alpha) F(T) \; \forall \alpha \in [0,1]$

14) Restricted Binary Additivity (RB. Addi):

$\forall S, T \in \Sigma^2$ with $h(S) = h(T)$ and $F_1(S) + F_2(S) =$

$F_1(T) + F_2(T)$, $F(V) = \frac{1}{2}[F(S) + F(T)]$ where

V = Comprehensive convex hull $\{S,T\}$ provided

$u(S) = \{\bar{A}(S)\}$ and $u(T) = \{\bar{A}(T)\}$.

We thus have the following theorem:

Theorem 2:

The only choice function on Σ^2 to satisfy PO, STC, SYM, RC.LIN and RB.Addi is $\bar{B}$.

Proof:

If F satisfies the assumptions then $F = \overline{B}$ is easily established along the lines of the relevant part of the proof in Theorem 1, by setting h(V) = (1,1) (permissible by STC) and by noting that Cases 3 and 4 can thus not arise. The other way is easy to check.

Q.E.D.

6 The Non-Symmetric Nash Choice Functions

The following assumption will be used in this section:

15) Independence of Irrelevant Expansions (IEE):

$\forall S \in \Sigma^2$ there exists a vector $p \in \mathbb{R}^2_+$ with $p_1 + p_2 = 1$ such that (a) $p.x = p.F(S)$ is the equation of a supporting line of S at F(S), (b) $\forall T \in \Sigma^2$ with $S \subset T$ and $p.x \leq p.F(S)$ $\forall x \in T$, we have $F(T) = F(S)$.

We are interested in a family of choice functions defined thus: Given $W = (W_1, W_2) \in \mathbb{R}^2_+$ with $W_1 + W_2 = 1$,

$$\text{Let} \quad F^W(S) = \underset{(x_1, x_2) \in S}{\operatorname{argmax}} \; x_1^{W_1} \; x_2^{W_2} \quad \text{if } W > 0$$

$$= (h_1(S), g_2(S)) \quad \text{if } W_1 = 1, W_2 = 0$$

$$= (g_1(S), h_2(S)) \quad \text{if } W_1 = 0, W_2 = 1$$

$\forall S \in \Sigma^2$. Here $(h_1(S), g_2(S))$ and $(g_1(S), h_2(S))$ belong to the Pareto optimal set of S whenever $S \in \Sigma^2$. The family $\{F^W / W \gg 0\}$ is called the family of nonsymmetric Nash choice functions. The family $\{F^W / W > 0\}$ is called the family of choice functions determined by a weighted heirarchy.

Example: W = (1,0)

Thus $F^{*}(S) = \left(h_1(S), g_2(S)\right) \forall S \in \Sigma^2$. But this F^{*} does not satisfy Independence of Irrelevant Expansions.

Take $S = \left\{x = (x_1, x_2) \in \mathbb{R}^2_+ / x_1^2 + x_2^2 \leq 1\right\}$.

Clearly $F^{*}(S) = (1,0)$. At (1,0), the unique supporting hyperplane in the definition of Assumption 3 is given by

p = (1,0). Now take $T = \left\{(x_1, x_2) \in \mathbb{R}^2_+ / x_1 \leq 1, x_2 \leq 1\right\}$.. Now T and S satisfy the conditions in Assumption 3, with p = (1,0). But $F^{*}(T) = (1,1) \neq F^{*}(S)$.

This example excludes the weighted hierarchy (1,0) as well as the weighted hierarchy (0,1) from the list of the possible candidates which could define a solution satisfying Assumption 3.

Hence the only possibilities are weighted hierarchies of the form $W >> 0$ i.e., a non-symmetric Nash choice function.

For the purpose of this section the following convention is adopted; Let $F: \Sigma^2 \rightarrow \mathbb{R}^2_+$ be a choice function satisfying Assumption 15. Given $S \in \Sigma^2$, let $p(F, S) = \{p \in \mathbb{R}^2_+ \setminus \{0\} / p$ satisfies the conditions of Assumption 15 for S}.

Theorem A: Let F be a choice function which satisfies PO, STC and IIE. Then F is a non-symmetric Nash choice function. Conversely, every non-symmetric Nash choice function satisfies PO, STC and IIE.

Proof: It is easy and somewhat routine to verify that every non-symmetric Nash bargaining solution satisfies PO, STC and IIE. Hence let us prove the converse. Hence assume that F is a choice function satisfying the desired assumptions.

Step 1: Let $\Delta_1 = \left\{(x_1, x_2) \in \mathbb{R}^2_+ / x_1 + x_2 \leq 1\right\}$. Then $F_i(\Delta_1) > 0$ for i = 1, 2.

Proof of Step 1: Let $S = \left\{(x_1, x_2) \in \mathbb{R}^2_+ / x_1^2 + x_2^2 \leq 1\right\}$. The example above show that (1,0) and (0,1) do not qualify as solutions for S which would satisfy assumption 15. Hence by (Pareto optimality, there exists $(y_1, y_2) > 0$, with $y_1^2 + y_2^2 = 1$ such that $F(S) = (y_1, y_2)$.

Let p.x = p.F(S) be the unique supporting line to S at F(S). Clearly $p_1 > 0$, $p_2 > 0$.

Let $T = \left\{(x_1, x_2) \in \mathbb{R}^2_+ / p.x \leq p.F(S)\right\}$.

By IIE $F(T) = F(S) = y = (y_1, y_2) \gg 0$. By Scale Transformation Covariance, $F(\Delta_1) = w > 0$.

Step 2: Let $S \in \Sigma^2$, such that $(h_1(S), h_2(S)) \in S$.

Then $F(S) = F^*(S) = (h_1(S), h_2(S))$.

Proof: Obvious by Assumption 1.

Step 3: Let $S \in \Sigma^2$ and $(h_1(S), h_2(S)) \in S$.

Then $(1,0), (0,1) \notin p(F,S)$.

Proof: Suppose towards a contradiction $(1,0) \in p\ (F,S)$ (the proof for $(0,1)$ is similar). Clearly $F(S) = (h_1(S), g_2(S))$ with $g_2(S) < h_2(S)$. Let $T = \{(x_1, x_2) \in \mathbb{R}^2_+ / x_i \leq h_i(S)\ i = 1,2\}$. S and T satisfy the conditions of IIE with $p = (1,0)$. But

$F(T) = (h_1(S), h_2(S))$ by PO; this contradicts IIE and proves Step 3.

Step 4: Let $S \in \Sigma^2$. If $S = \{(x_1, x_2) \in \mathbb{R}^2_+ / x_i \leq h_i(S),\ i = 1,2\}$,

then Step 2 establishes that F(S) is a non-symmetric Nash Solution for S.

Hence assume $(h_1(S), h_2(S)) \notin S$. By Step 3 and PO, $(1,0)$, $(0,1)$ $\notin p\ (F,S)$ irrespective of whether $F_1(S) = h_1(S)$ or $F_2(S) = h_2(S)$ or otherwise.

Let $p \in p\ (F,S)$ where by STC we may assume $F_i(S) = 1,\ i = 1,\ 2$.

Let $T = \{(x_1, x_2) \in \mathbb{R}^2_+ / p_1 x_1 \le p_2 x_2\}$.

Since $F(\Delta_1) = w$, by STC, $F(T) = F^w\ (T)$

But this T and S satisfy the condition of IIE assumption. Hence $F(S) = F(T) = F^w\ (T) = F^w\ (S)$.

Thus $F(S) = F^w(S)$. Q.E.D.

Our next objective is to invoke the assumption of weak independence of irrelevant expansions defined in Peters (1986b) and establish a result similar to his.

6) **Weak Independence of Irrelevant Expansions (WIEE):**

$\forall\ S \in \Sigma^2$ there exists a vector $p \in \mathbb{R}^2_+$ with $p_1 + p_2 = 1$ such that:

a) $p.x = p.F(S)$ is the equation of a supporting line of S at $F(S)$;

b) $\forall\ T \in \Sigma^2$ with $S \subset T$ and $p.x \le p.F(S)\ \forall\ x \in T$, we have $F(S) \le F(T)$.

Notice that Assumption 15 implies Assumption 16. Hence the non-symmetric Nash choice functions satisfy Assumption 16 as well.

7) Family of Choice Functions Determined by a Weighted Hierarchy:

For the purpose of this section, the following convention is adopted: Let $F: \Sigma^2 \to \mathbb{R}^2_+$ be a choice function satisfying Assumption 16. Given $S \in \Sigma^2$, let $p(F,S) = \{p \in \mathbb{R}^2_+ \setminus \{0\} / p$ satisfies the conditions of Assumption 16 for S$\}$. For Lemmas 1, 2 and 3 below we assume that F satisfies PO, STC and WIIE.

<u>Lemma 1:</u> If $(1,0) \in p(F,S)$ for some $S \in \Sigma^2$ with $S \neq Comv\{h(S)\}$ then $F(T) = (g_1(T), h_2(T)) \; \forall \, T \in \Sigma^2 \setminus \{ a\Delta_1 / a > 0 \}$.

<u>Proof</u>:- Suppose there exists $T \in \Sigma^2 \setminus \{ a\Delta_1 / a > 0 \}$ such that $F(T) \neq (g_1(T), h_2(T))$.

Clearly (a) $(1,0) \notin p(F,T)$

(b) $T \neq Comv\{h(T)\}$.

Now, $(1,0) \in p(F,S)$, implies by PO that

$F(S) = (g_1(S), h_2(S))$

Let $V = Comv\{u,v\}$, where

$u_2 = h_2(S)$, $v_1 = h_1(S)$, $u_1 > g_1(S)$, $v_2 > h_2(S)$,

$u_2 > v_2$, $u_1 < v_1$, $u >> 0$, $v >> 0$.

Such a V exists since $S \neq Comv\{h(S)\}$

By PO and WIIE, $F(V) = u = (g_1(V), h_2(V))$

By STC, $F(V) = u = (g_1(V), h_2(V)) \; \forall \, V \in \Sigma^2$ with

$V = Comv\{u,v\}$, $u >> 0$, $v >> 0$, $u_2 > v_2$, $u_1 < v_1$.

Now, $T \in \Sigma^2 \setminus \{ a\Delta_1 / a > 0 \}$, $(1,0) \notin p(F,T)$ implies that there exists V as above (i.e. $V = Comv\{u,v\}$ such that

$F(T) = v$ if $(0,1) \in p(F,T)$

$\neq u, v$ if $(0,1) \notin p(F,T)$

with $F(T) \in p(V)$,

$T \subset V$.

By WIIE, $F(V) = F(T) \neq u$

This contradiction establishes the lemma.

<u>Q.E.D.</u>

Lemma 2:- If $(1,0) \in p\ (F,S)$ for some $S \in \Sigma^2$ with $S \neq \text{Comv}\ \{h(S))\}$ then $F(T) = (g_1\ (T),\ h_2\ (T))\ \forall\ T \in \Sigma^2$.

Proof:- Given Lemma 1 above and by appealing to STC, it is enough to show that $F\ (\Delta_1) = (0,1)$

Let $T = \{\ x \in \Delta_1 / x_1 \leq ½\ \}$

$T \in \Sigma^2 \setminus \{\ a\Delta_1 / a >> 0\ \}$.

By Lemma 1, $F(T) = (0,1)$.

By WIIE (since $T \subset \Delta_1$, with the conditions of WIIE being trivially satisfied for T and Δ_1 at $(0,1)$), $F(\Delta_1) = (0,1)$.

Q.E.D.

Lemma 3: If $(0,1) \in p\ (F,S)$ for some $S \in \Sigma^2$ with $S \neq \text{Comv}\ \{h(S)\}$ then $F(T) = (h_1\ (T),\ g_2\ (T))\ \forall\ T \in \Sigma^2$.

Proof:- Similar to above (i.e. Lemmas 1 and 2).

Lemma 4: Suppose $(1,0)$, $(0,1) \notin p(F,V)$ whenever $V \in \Sigma^2$ $V \neq \text{Comv}\ \{h(v)\}$. If F satisfies PO, STC and WIIE, then F is a non-symmetric Nash bargaining choice function.

Proof: Let $F(\ _1$ w 0 since $(1,0)$, $(0,1)$ $p(F,\ _1)$

Thus $F(a\Delta_1) = F^w(a\Delta_1)\ \forall\ a \in \mathbb{R}^2_{++}$

Now let $S \in \Sigma^2$ $S \neq \text{Comv}\{h(S)\}$. Then $\forall\ p \in p\ (F,S)$, $p \gg 0$

Let $T = \{(x_1 + x_2) \in \mathbb{R}^2_+ / p_1 x_1 + p_2 x_2 \leq p_1 F_1(S) + p_2 F_2(S)\}$.

Clearly $F(T) = F^w\ (T)$ and $F(T) = F(S)$ the latter by PO and WIIE. Thus $F(S) = F^w\ (T)$. Since $F^w(T) = F^w\ (S)$, we have the desired result.

Q.E.D.

Note: By Assumption 2, if $F(\Delta_1) = F^w(\Delta_1)$ for some $W > 0$, then $F(a\Delta_1) = F^w(a\Delta_1)\ \forall\ a \in R^2_{++}$ and for the same W. Since $F(\Delta_1)$ is always equal to some $F^{w}(\Delta_1)$ with $W > 0$, $F(a\Delta_1)$ is always equal to $F^{w}(a\Delta_1)\ \forall\ a \epsilon R^2_{++}$ for some fixed $W > 0$, $W_1 + W_2 = 1$.

As a consequence of the above lemmas we have the following theorem.

<u>Theorem 4:</u> Let F be a choice function on Σ^2 which satisfies Assumptions 2, 3 and 16. Then $F = F^{w}$ for some $W = (W_2, W_2) > 0$ with $W_1 + W_2 = 1$.

Conversely, any choice function F^{w} with $W > 0$, $W_1 + W_2 = 1$ satisfies PO, STC and WIIE.

Appendix A

In this appendix and in view of Remark (2) (after Theorem 1), we prove an axiomatic characterization of the egalitarian choice function using the superadditivity axiom. We invoke the following two assumptions as well.

Strong Individual Rationality (SIR):

$F(S) > 0 \; \forall \; S \in \Sigma^2$

Continuity (CONT):

If $\{S^k\}$ be a sequence in Σ^2 converging to $S \in \Sigma^2$ in the Hausdorff topology, then $\lim_{k \to \infty} F(S^k) = F(S)$.

We now prove the following theorem:

Theorem:

The only choice function on Σ^2 to satisfy SIR, WPO, SYM, NIIA, S.Addi and CONT is the egalitarian choice function E defined as follows:

$\forall \; S \in \Sigma^2, \; E(S) = (\bar{t}, \bar{t})$, where $\bar{t} = \max\{t / (t,t) \in S\}$.

To prove this theorem we use the following lemma:

Lemma:

Under the hypothesis of the theorem, $F(T) \geq E(T) \; \forall \; T \in \Sigma^2$ of the form $T = \{x \in \mathbb{R}^2_+ / x \leq a\}$ for some $a > 0$.

Proof:

If $a = (a_1, a_2)$ with $a_1 = a_2$, then $F(T) = E(T)$ by WPO and SYM.

Hence suppose W.l.o.g. $a_1 > a_2$.

Thus $E(T) = (a_2, a_2)$

Let $b(\epsilon) = (1 - \epsilon)\, a_2$ for $0 < \epsilon < 1$.

$$T(\epsilon) = \{x \epsilon \mathbb{R}^2_+ / x \leq (b(\epsilon), b(\epsilon))\}$$

$$U(\epsilon) = \{x - (b(\epsilon), b(\epsilon)) / x \geq (b(\epsilon), b(\epsilon))\ x \epsilon T\}.$$

Then $T = T(\epsilon) + U(\epsilon)\ \forall\ 0 < \epsilon < 1$.

$$\therefore F(T) \geq F(T(\epsilon)) = (b(\epsilon), b(\epsilon))\ \forall\ 0 < \epsilon < 1.$$

Taking limits as $\epsilon \to 0$, we get $F(T) \geq E(T)$. Q.E.D.

Proof of Theorem:

That E satisfies the above properties is clear. Thus let us assume F satisfies the above properties and towards a contradiction assume that there exists $S \epsilon \Sigma^2$ such that $F(S) \neq E(S)$. To begin with assume $E(S)\ \epsilon\ P(S)$. The proof is completed by appealing to CONT.

Let T = Comprehensive convex hull $\{F(S)\}$

By NIIA, $F(T) = F(S)$

By Lemma above $F(T) \geq E(T)$

Clearly $F(T) \neq E(T)$ for then $F(S) = E(S)$

Without loss of generality assume $F_1(T) > E_1(T)$

Since $E(T)\ \epsilon\ W(T)$, $F_2(T) = E_2(T)$

Let T' = Comprehensive convex hull $\{E(T)\}$

$$F(T') = E(T') = E(T)$$

Let $U = \{x - E(T)\ \epsilon\ \mathbb{R}^2_+ / x \epsilon S\}$

$U \epsilon \Sigma^2$, since $E(S)\ \epsilon\ P(S)$

$T' + U \subset S$ and $F(S) = F(T)\ \epsilon\ U + T'$

By NIIA, $F(T' + U) = F(S) = F(T)$

But $F(T' + U) \geq F(T') + F(U)$ by S. Addi.

i.e. $F(T) \geq E(T) + F(U)$

By SIR, $F(U) \gg 0$

$\therefore F(T) \gg E(T)$

Contradicting $F_2(T) = E_2(T)$. Q.E.D.

In the above proof we invoke the Nash's Independence of irrelevant Alternatives Assumption, which sets the egalitarian choice function apart both from the choice function of Perles and Maschler [1981] and the choice function that we define in this paper.

Further since, SIR + HOM + NIIA → WPO, the following collary is immediate:

Corollary:

The only choice function on Σ^2 to satisfy SIR, HOM, NIIA, S. Addi, SYM and CONT is the egalitarian choice function.

Appendix B

The purpose of this appendix is to establish a replication invariance property for the additive choice function. Replication invariance for the egalitarian choice function has been established in Lahiri [1996] and for the relative egalitarian and Nash choice functions in references contained in the same paper. In order to establish the replication invariance property we need the following framework.

Let $n \in \mathbf{N}$ and $\mathbf{R}^n_+$ denote the non-negative orthant of n dimensional Euclidean space. A choice problem in $\mathbf{R}^n_+$ (often called an n - dimensional choice problem) is a non-empty set S in $\mathbf{R}^n_+$ satisfying the following properties:

i) $0 \in S$

ii) S is compact, convex and comprehensive (i.e. $0 \le x \le y \in S \rightarrow x \in S$)

iii) $\exists\, x \in S$ with $x \gg 0$.

Let Σ^n denote the class of all n dimensional choice problems. We shall be interested in a subclass of Σ^n in what follows.

Given $S \in \Sigma^n$,

$$\text{let } u(S) = \left\{ x \in S / \sum_{i=1}^{n} x_i \ge \sum_{i=1}^{n} y_i \ \forall\, y \in S,\ y = (y_i)_{i=1}^{n} \right\}$$

We shall be interested in the following subclass of Σ^n denoted B^n : $S \in B^n$ if and only if the compact convex set u(S) has a finite number of extreme points. Let e(S) denote the set of extreme points of u(S), whenever $S \in B^n$ and let $|e(S)|$ denote its cardinality. The <u>additive choice function</u> $\bar{A}: B^n \rightarrow \mathbf{R}^n_+$ is defined as follows:

$$\bar{A}(S) = \frac{1}{|e(S)|} \sum_{x \in e(S)} x, \text{ whenever } S \in B^n.$$

Let $S \in \Sigma^2$ be given, as well as natural numbers m,l. Let $I_m = \{1, 2, \ldots, m\}$ and $J_l = \{m+1, \ldots, m+l\}$. For a pair $(i,j) \in I_m \times J_l$, let

$$S_{i,j} = \left\{x \in \mathbb{R}^{m+l} / \exists (x_1', x_2') \in S \text{ with } x_i = x_1', x_j = x_2', x_k = 0 \text{ if } k \neq i,j\right\}.$$

The Thomson (m,l) replication of S is defined as $S^{m,l} = Conv\left\{S_{ij} / (i,j) \in I_m \times J_l\right\}$. Clearly $S^{m,l} \in B^{m+l}$. Indeed, if x^{ij} denotes an element of $S_{i,j}$, then the extreme point of u(S) are $\left\{a^{ij}(S),\ b^{ij}(S),\ (i,j) \in I_m \times J_l\right\}$ where $a_i^{ij}(S) = a_1(S), a_j^{ij}(S) = a_2(S),\ a_k^{ij}(S) = 0$ if $k \neq i,j$; $b_i^{ij}(S) = b_1(S), b_j^{ij}(S) = b_2(S),\ b_k^{ij}(S) = 0$ if $k \neq i,j$. Thus

$$\bar{A}(S^{m,l}) = \frac{1}{2ml}\left[\sum_{(i,j) \in I_m \times J_l} a^{ij}(S) + \sum_{(i,j) \in I_m \times J_l} b^{ij}(S)\right].$$

<u>Theorem:</u>

In the above frame work, $m\bar{A}_i(Sm,l) = \bar{A}_1(S)\ \forall\, i \in I_m$ and $l\bar{A}_j(S^{m,l}) = \bar{A}_2(S)\ \forall\, j \in J_l$

<u>Proof:</u>

Let $(c,d) = \bar{A}(S)$.

Thus $(c,d) = \frac{m}{m+l}\, a(S) + \frac{l}{m+l}\, b(S)$

Now $\bar{A}_k(S^{m,l}) = \frac{1}{2ml}\left[\sum_{j \in J_l} a_k^{k,j}(S) + \sum_{j \in J_l} b_k^{kj}(S)\right]$ if $k \in I_m$

$$= \frac{1}{2ml}\left[\sum_{i\in I_m} a_k^{ik}(S) + \sum_{i\in I_m} b_k^{ik}(S)\right] \quad if \quad k\in J_1$$

$$\therefore \bar{A}_k\left(S^{m,1}\right) = \left[1a_1(S) + 1b_1(S)\right] \quad if \quad k\in I_m$$

$$= \frac{1}{2ml}\left[ma_2(S) + mb_2(S)\right] \quad if \quad k\in J_1$$

$$\therefore \bar{A}_k\left(S^{m,1}\right) = \frac{1}{2m}\left[a_1(S) + b_1(S)\right] \quad if \quad k\in I_m$$

$$= \frac{1}{2m}\left[a_2(S) + b_2(S)\right] \quad if \quad k\in J_1$$

Thus, $m\bar{A}_k\left(S^{m,1}\right) = A_1(S) \; \forall \; k\in I_m$

$1\bar{A}_k\left(S^{m,1}\right) = A_2(S) \; \forall \; k\in J_1$. Q.E.D.

Let us show that, $\sum_{k\in I_m\cup J_1} \bar{A}_k\left(S^{m,1}\right) \geq \sum_{k\in I_m\cup J_1}$

$\forall \, x\in = \left(x_k\right)_{k\in I_m\cup J_1} \in S^{m,1}$

Let $(c,d) = \frac{1}{2}[a(S) + b(S)] \in u(S)$.

Thus $c + d \geq x_1' + x_2' \; \forall\left(x_1', x_2'\right)\in S$. Thus if x^{ij} denotes a vector in S^{ij}, then $c + d \geq x_i^{ij} + x_j^{ij}$.

Now, let $y\in S^{m,1}$. Then, there exists $\mu_{ij} \geq 0, (i,j) \in I_m \times J_1$ such that $y\in \leq \sum_{(i,j)\in I_m\times J_1} \mu_{ij}\, x^{ij}$ for some x^{ij}, $(i,j) \in I_m \times J_1$, and $\sum_{(ij)\in I_m\times J_1} \mu_{ij} = 1$.

$$\therefore y_k \leq \sum_{j\in J_1} \mu_{kj}\, x_k^{kj} \quad if \quad k\in I_m$$

$$y_k \leq \sum_{i\in I_m} \mu_{ik}\, x_k^{ik} \quad if \quad k\in J_1$$

$$\therefore \sum_{k \in I_m} y_k + \sum_{k \in J_l} y_k \le \sum_{k \in I_m} \sum_{j \in J_l} \mu_{kj} x_k^{kj} + \sum_{k \in J_l} \sum_{i \in I_m} \mu_{ik} x_k^{ik}$$

$$= \sum_{(i,j) \in I_m \times J_l} \mu_{ij} x_i^{ij} + \sum_{(i,j) \in I_m \times J_l} \mu_{ij} x_j^{ij}$$

$$\sum_{(ij) \in I_m \times J_l} \mu_{ij} \left[x_i^{ij} + x_j^{ij}\right] \le c + d$$

$$= \sum_{i \in I_m} \frac{c}{m} + \sum_{j \in J_l} \frac{d}{l}$$

$$\sum_{k \in I_m \cup J_l} \bar{A}_k(s^{m,l}).$$

This establishes the bonafides of the extension of $\bar{A}$ from Σ^2 to B^n as introduced in this appendix.

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