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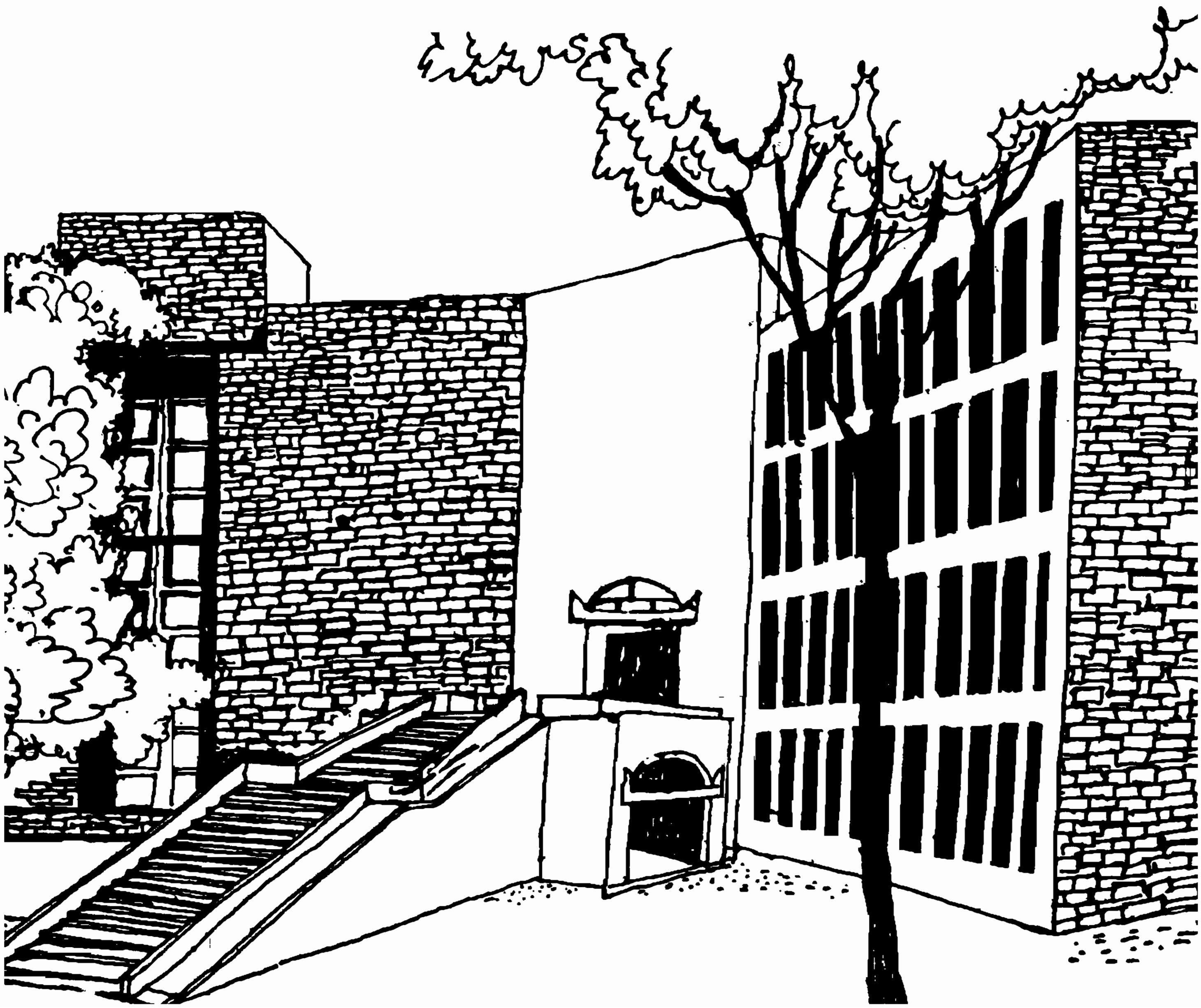

A REDUCED GAME PROPERTY FOR THE PROPORTIONAL SOLUTION FOR CLAIMS PROBLEMS

By

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I would like to thank Pankaj Chandra for introducing me to the supply chain management interpretation of a claims problem and to R Sridharan for subsequent discussions. They also raised the very poignant issue of incentives for truthfull revelation of demands by the retailers, which is addressed only marginally in this paper, simply because it is my express belief (not shared by most others in the profession) that such problems naturally lead us into the domain of ethics which is far removed from the issue of resource allocation. Therefore and otherwise Pankaj and Sridhar are in no way responsible for whatever deficiencies and errors this paper still contains.

Abstract

The problem we discuss in this paper is one of allocating a homogeneous, divisible good among a group of claimants in a way that is perceived as just or fair. A solution to such a problem is allocating the good in proportion to the claims. We use a reduced game property to axiomatically characterize this solution. The model is interpreted as a distributor allocating a good amongst several retailers when demand exceeds supply.

1. Introduction: The problem we discuss in this paper is one of allocating a homogeneous, divisible good among a group of claimants in a way that is perceived as just or fair. The genesis of such a problem is in the Babylonian Talmud, an analytical treatment of which can be found in O'Neill [1982] followed by Aumann and Maschler [1985]. Game theoretic treatments of two prominent solutions to such problems can be found in Dagan and Volij (1993). Further axiomatic characterizations of some solutions can be found in Dagan (1996).

To dispel the impression that such problems can be used to resolve only bankruptcy problems in ancient law, a series of papers written by Young [1987a, 1987b, 1988], underscored its importance in obtaining principles of just taxation. A lucid rendition of the problem can be found in Young [1993]. However, there is yet another application which merits mentioning: consider the framework of supply chain management where a factory supplies a product to a distributor, who in turn supplies the commodity to a finite member of retail outlets. If the aggregate demand of the retailers is less than or equal to the amount available with the distributor, the solution reduces to the trivial one of providing each retailer with what he/she requires. If on the other hand the aggregate demand of the retailers exceeds the amount available with the distributor, the problem of allocating the commodity among the retailers becomes non-trivial and the body of literature which originated from O'Neill's paper can be used to shed light divided. We assume throughout that $\sum_{i \in Q} c_i > M$. Let $\mathscr{E}^Q$ be the set of all claims problems for Q and $\mathscr{E} = \bigcup_{Q \in Z} \mathscr{E}^Q$. Let $X = \bigcup_{Q \in Z} \mathbf{R}_+^Q$

Given $Q \in Z$, and $(c, M) \in \mathscr{E}^Q$, an allocation is a vector $x \in R_+^Q$ such that $\sum_{i \in Q} x_i = M$ and $x_i \le c_i \forall i \in Q$.

A <u>solution</u> is a function $F: \mathscr{E} \to X$ such that $\forall (c, M) \in \mathscr{E}$, , F (c,M) is an allocation for (c, M).

The <u>proportional</u> solution $P: \mathscr{E} \to X$ is defined thus:

$\forall Q \in Z, \forall (c, M) \in \mathscr{E}^Q, P(c, M) = \theta(c, M)\, c,$ *where* $\theta(c, M) > 0$ is chosen to satisfy $\sum_{i \in Q} \theta(c, M)\, c_i = M$

Clearly, $\theta(c, M) < 1,$ *since* $\sum_{i \in Q} c_i > M$

Thus $P_i(c, M) < c_i \; \forall i \in Q$

We are interested in the following property:

<u>Reduced Game Property (RGP)</u>:

Given $Q \in Z, |Q| \geq 2$ and $(c,M) \in \mathscr{E}^Q$, let $x = F(c,M)$. Let $\phi \neq L \subset Q$ with $x_L > 0$ and $y = F(c_L, M)$, where $c_L = (c_i)_{i \in L}$. Let $\dot{M} = \sum_{i \in L} x_i$. Then $F_L(c,M) = \frac{\dot{M}}{M} F(c_L, M)$.

A weaker version of the above property known as the Weak Reduced Game Property, is simply the same statement with cardinality of L equal to two i.e. $|L| = 2$. It is easy to see that RGP implies Weak RGP.

In the rest of the paper we prove that the proportional solution satisfies RGP and that the only solution to satisfy Weak RGP is the proportional solution provided the solution agrees with the proportional solution for all two agent problems.

The Main Results:

Theorem 1: P satisfies RGP

Proof: Let

$Q \in Z$, $|Q| \geq 2$, $\phi \neq L \subset Q$, $(c,M) \in \mathscr{E}^Q$ and $x = P(c,M)$. $x_L > 0$ and let $y = P(c_L, M)$.

Thus $x = \theta\,(c,M)\,c$ and $y = \theta\,(c_L, M)\;c_L$, $\dot{M} = \theta\,(c,M)\;\sum_{i \in L} c_i$

$$\therefore\; x_L = \theta\,(c,M)\,c_L = \frac{\dot{M}}{\sum_{i \in L} c_i}\,c_L$$

on the problem.

A very common solution to this problem is allocating the divisible commodity in proportion to the claims, so that the entire amount is exhausted. There have been several axiomatic characterizations of the proportional solution, available in the literature, some of which have been cited above. Our purpose in this paper is to characterize the proportional solution for such claims problems, using a reduced game property. A reduced game property, due to Peters, Tijs and Zarzuelo [1994], which has been used by them to characterize the relative egalitarian choice function for choice problems (and also by Lahiri[1996], to characterize the egalitarian choice function) is modified to be meaningful in the present context. It is shown in this paper, that the only solution to satisfy this reduced game property (and even a weaker version of the same) is the proportional solution provided another relatively mild condition is imposed.

2. The Model: Let **N** denote the set of potential claimants whether finite or infinite. If infinite it will be assumed to be the set of natural numbers $\mathbf{N}$; if finite it will be assumed to be the set of first n natural numbers for some $n \in \mathbf{N}$. Let Z denote the set of all non-empty, finite subsets of **N**.

A claims problem for $Q \in Z$ is an ordered pair (c, M) $\in \mathbb{R}^{Q}_{++} \times \mathbb{R}_{++}$ where for $\forall i \in Q$, c_i denotes the claim of claimant i and, M is the total amount to be

$$= \frac{\dot{M}}{M} \frac{M}{\sum_{i \in L} c_L} c_L$$

$$= \frac{\dot{M}}{M} \theta \ (c_L, M)\, c_L$$

$$= \frac{\dot{M}}{M} y$$

Since $\sum_{i \in L} y_i = \theta \ (c_L, M) \sum_{i \in L} c_i = M$

$\rightarrow \theta \ (c_L, M) = M / \sum_{i \in L} c_i$

<u>Theorem 2</u>: Suppose $F: \mathcal{E} \rightarrow X$ is such that $\forall\, Q \epsilon Z$ with $|Q| = 2$ and all $(c, M) \epsilon \mathcal{E}^Q$ we have $F(c, M) = P(c, M)$. If F satisfies Weak RGP, then $F = P$ on $\mathcal{E}$.

<u>Proof</u>: Let $Q \epsilon Z$. If $|Q| = 1$ or 2 there is nothing to prove. Hence assume $|Q| > 2$. Without loss of generality assume $Q = \{1, \ldots, n\}$ for some $n \epsilon \mathbf{N}$ with $n > 2$.

Let $(c, M) \epsilon \mathcal{E}^Q$ and $X = F(c, M)$. We have to show that $X = P(c; M)$. Consider $i \varepsilon Q$, $i \neq 1$. By the hypothesis of the theorem, $F_1(c_1, c_i; M) = \frac{c_1}{c_1 + c_i} M$.

By Weak RGP, $X_1 = \frac{X_1+X_i}{M} \frac{c_1}{c_1+c_i} M$

i.e. $X_1(c_1+c_i) = c_1(X_1+X_i)$ i.e. $X_1 c_i = c_1 X_i$

$\therefore X_1 \sum_{i\neq 1} c_i = c_1 \sum_{i\neq 1} x_i$

Adding $c_1 x_1$ to both sides, we get

$X_1 \sum_{i\epsilon Q} c_i = c_1 \sum_{i\epsilon Q} x_i = c_1 M.$

$\therefore X_1 = \frac{c_1}{\sum_{i\epsilon Q} c_i} M$

Since instead of 1 we could have chosen any $j\varepsilon Q$ and obtained,

$X_j = \frac{c_j}{\sum_{i\epsilon Q} c_i} M,$

We get that F(c,M) = P(c,M).

This proves the theorem.

<u>Q.E.D.</u>

This theorem essentially defines the proportional solution uniquely on the class of all claims problems, modulo the restriction that it is already known that for all two dimensional problem it has the explicit functional representation of the proportional solution. Hence the only problem is to characterise the proportional solution for two dimensional problems.

It may be argued that for several types of problems, notably the kind envisaged by the supply chain management problem, the proportional rule is the natural one to apply for two agents (i.e. two retailer) situations. Indeed, if the distributor is impartial as far as retailers go, then what could be more natural than dividing an amount between them in proportion to their demands (which in effect is a proxy for the segmented market demands). However, this reasoning is a justification for applying the proportional rule not merely in the two agent situation, but for situations consisting of any finite number of agents. Thus, inspite of the fact that the given reasoning is very convincing, from the standpoint of the present paper it is insufficient, since it is not amenable to any analytical expression other than the direct one. Put simply, in this paper we want to derive the proportional solution, not define it. Thus we suggest the following property:

Restricted Scale Invariance for Two Agents:

$\forall i, j \in N, i \neq j, \forall (c, M), (\acute{c}; M) \in \mathcal{E}^{(i,j)}$ if $c_i + c_j = \acute{c}_i + \acute{c}_j$, *then* $F_i(\acute{c};M) = \frac{\acute{c}_i}{c_i} F_i(c,M)$

and $F_j(\acute{c};M) = \frac{\acute{c}_j}{c_j} F_j(c;M)$.

The property Restricted Scale Invariance for Two Agents is fairly strong; it says that given two hypothetical situations where two retailers place different demands with the distributor, if it turns out that the aggregate demand remains the same, then for each retailer, the ratio of the awards should be equal to the ratio of the demands. Observe, this covers the situation where the retailers swap their demands i.e. a simple permutation. It is instructive to note that in the sequel no additional symmetry assumption is required to characterise the proportional solution for two agent problems.

<u>Lemma 1</u>: Let $|N| \geq 2$. *Suppose* $F: \mathcal{E} \to X$ satisfy Restricted Scale Invariance for Two Agents. Then $\forall\, Q \in Z$ *with* $|Q| = 2$, $\forall\, (c,M) \in \mathcal{E}^Q$, $F(c,M) = P(c,M)$

<u>Proof of Lemma 1:</u> Let $Q = \{i, j\}$, $i,j \in N$ and suppose

$(x_i, x_j) = (F_i(c_i, c_j; M), F_j(c_i, c_j, M))$ *where* $(c_i, c_j; M) \in \mathcal{E}^Q$.

By Restricted Scale Invariance for Two Agents, there exists functions $f_i : \mathbb{R}^2_{++} \to \mathbb{R}_{++}$ and $f_j : \mathbb{R}^2_{++} \to \mathbb{R}_{++}$ such that

$x_i = c_i f_i(c_i + c_j, M) \quad x_j = c_j f_j(c_i + c_j, M)$

Let d = $c_i + c_j$. *Then* $c_i f_i(d, M) + c_j f_j(d, M) = M$

$\forall c_i, c_j > o$ *such that* $c_i + c_j = d$. *Let* $c_i = c_j = d/2$

$\therefore f_i(d, M) + f_i(d, M) = 2M/d$ Suppose towards a contradiction that for some

$0 < M < d$, $f_i(d, M) = \theta \frac{M}{d} > M/d$ (: *the case where* $f_i(d, M) = \theta \frac{M}{d} < \frac{M}{d}$ *is similar since in that case* $f_j(d, M) > M/d$).

$$\therefore c_i \theta \frac{M}{d} + c_j (2 - \theta) \frac{M}{d} = M \forall c_i, c_j > 0$$

with $c_i + c_j = d$.

$\therefore \theta c_i + (2 - \theta) c_j = c_i + c_j$ *i.e.* $(\theta - 1) c_i = (\theta - 1) c_j$

$\rightarrow c_i = c_j$ *since* $\theta \neq 1$.

But this is a contradiction since $f_i(d, M)$ is independent of

c_i *and* c_j. *Thus* $f_i(d, M) = \frac{M}{d} = f_j(d, M)$.

Q.E.D.

As a corollary to Theorem 1, 2 and Lemma 1 we have the following:

Corollary 1: The only solution on $\mathcal{E}$ to satisfy RGP, and Restricted Scale Invariance for Two Agents is P.

Corollary 2: The only solution on $\mathcal{E}$ to satisfy Weak RGP, and Restricted Scale Invariance for Two Agents is P.

In the introduction we have referred to the last condition as relatively mild. The assumption of Restricted Scale Invariance for Two Agents is mild when viewed as a requirement applicable only for two dimensional problems, whereas our claims problems can entertain arbitrary finite number of agents.

4. Incentives to Misrepresent Demand:

The entire framework above, when adapted to the supply chain management situation is riddled with the possibility of retailers misrepresenting their demands since they operate in a situation of rationing. The possibility of the distributor knowing the true demands, though realistic, is contrary to the spirit of decentralization in which this paper has been conceived. Thus retailers do benefit by inflating their demands.

Let us assume that the retailers inflate their demand uniformly and multiplicatively i.e. in each succeding period the previous demand is multiplied by a constant say $x > 1$, x being the same for all retailers. In this case the proportional rule remains intact and inviolable.

On the other hand if in each succeeding period they inflate their demand uniformly but additively i.e. by adding $x > 0$, then the proportional rule converges to the rule which allocates the good equally among the retailers.

There are a host of other possibilities open which leads to a distortion in the proportional rule and which may be amenable to a separate analysis. We leave such an analysis as an open problem for the interested reader.

It should be pointed out at this juncture, that the possibility of misrepresenting demands may defeat the purpose of rationing when there are chronic shortages. However, if shortages are unforeseen (which is tantamount to the retailers being unaware of the true supply), then the main analysis outlined in this paper carries through in letter and spirit.

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