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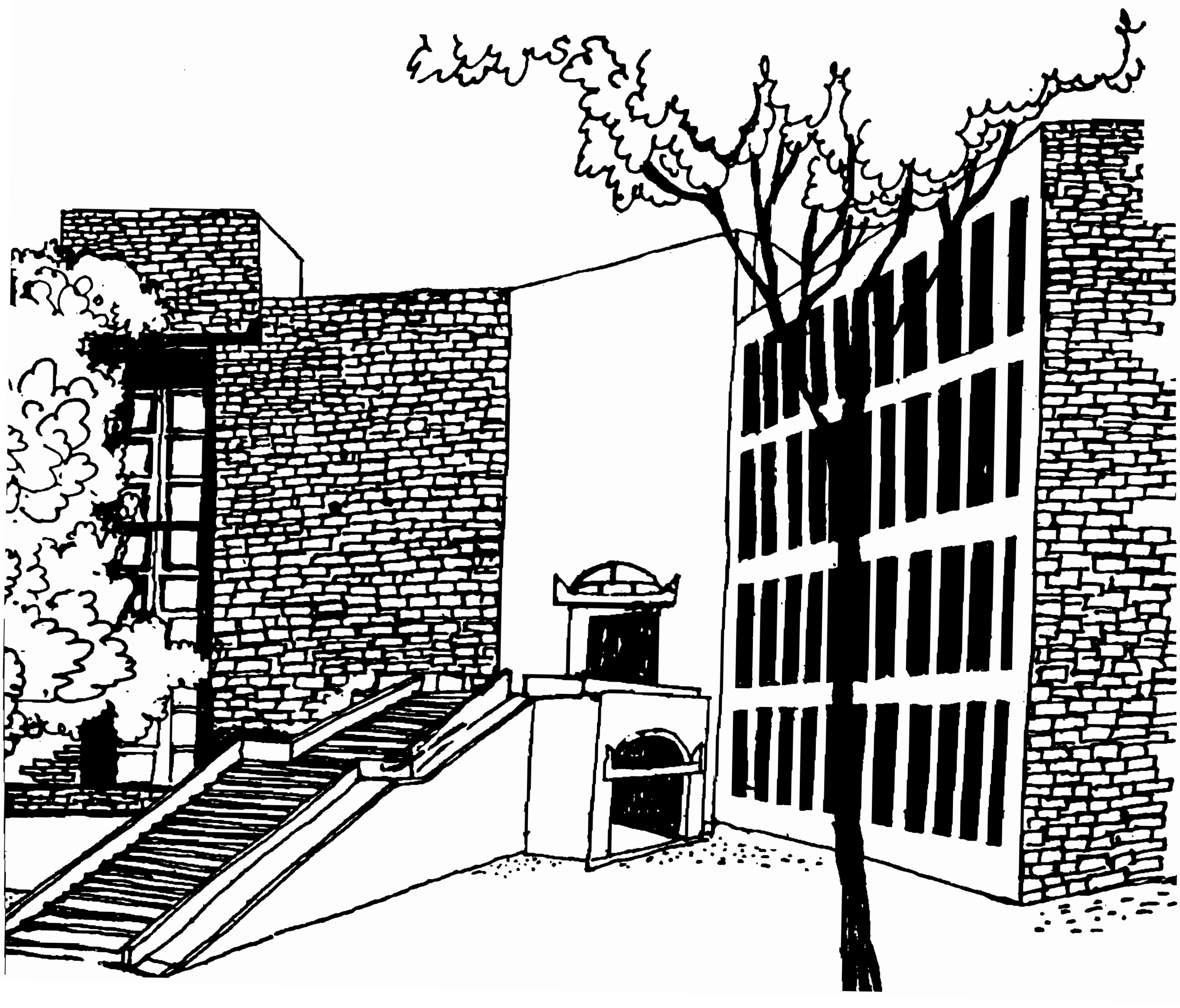

PARTIAL ISSUE MONOTONICITY

By

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Abstract

In this paper we propose and characterize a class of new solutions to bargaining problems which uses a reference point and satisfies a property known as partial issue monotonicity..

1. Introduction :- Motivated by the generalization of Nash's Axiom of Irrelevant Alternative in Thomson (1981), we propose a variant of restricted issue monotonicity (see Moulin (1988)) (which is used in characterizing the relative egalitarian solution due to Kalai and Smorodinsky (1975)). This variant provides a whole new class of solutions which satisfies a property known as partial issue monotonicity.

We use the concept of a reference function, which was defined for the first time in Thomson (1981). This reference function, plays the same role in our context, as the utopia point plays in the definition of the relative egalitarian solution. An example of a reference function is provided, and this example is one of a host of others (not discussed in the paper) which could meet our purpose. The formulation in this paper avoids the Independence of Irrelevant Alternative assumption due to Nash (1950).

2. The Framework :- Given are society $N=\{1,\ldots,n\}$ consisting of n agents and a set Σ of subsets of $\mathbb{R}^N_+$. A <u>social choice function</u> on (N,Σ) is a mapping $\varphi:\Sigma\to\mathbb{R}^N_+$ such that $\varphi(S)\in S\ \forall S\in\Sigma$. Σ is called the <u>'domain'</u>.

In order to facilitate exposition and without sacrificing crucial details, we work with a domain Σ_0 which satisfies the following property :

$S\in\Sigma_0 \Leftrightarrow$ S is a <u>convex, compact, comprehensive</u> ($u\in S$, $v\in\mathbb{R}^N_+$, $v\leq u \Rightarrow v\in S$) subset of $\mathbb{R}^N_+$, satisfying <u>minimal transferability</u> ($\forall u\in S$, all $i\in N$: $(u_i>0) \Rightarrow \exists v\in S$, $v_i<u_i$ and $v_j>u_j$, all $j\neq i$).

Convexity, compactness and comprehensiveness are common assumptions for domains in axiomatic models of bargaining. Minimal transferability is introduced here in order to erase distinctions between weak Pareto optimality and Pareto optimality. This definitely helps in exposition, although it is not necessary to the extent that the other properties are.

We next define a reference function on the domain Σ_0. A reference function on the domain Σ_0, is a function $\bar{u}:\Sigma_0\to\mathbb{R}^N_+$ satisfies the following property:

$\bar{u}(S)\gg 0, \forall S\in\Sigma_0$, $S\neq\{0\}$.

We assume that our reference function $\bar{u}:\Sigma_0 \to \mathbb{R}^N_+$ satisfies the following two properties:

(i) Anonymity :- For any permutation σ of N and any vector $u \in \mathbb{R}^N_+$, write $\sigma(u)$ for the vector where $\sigma(u)_i = u_{\sigma(i)}$ $\forall i \in N$. Then $\forall S \in \Sigma_0$: $\bar{u}(\sigma(S)) = \sigma(\bar{u}(S))$ where $\sigma(S) = \{\sigma(u)/u \in S\}$.

(ii) Scale Independence :- For all $\alpha \in \mathbb{R}_{++}$, all $S \in \Sigma_0$: $\bar{u}(\alpha.S) = \alpha.\bar{u}(S)$ with $\alpha.u \equiv (\alpha_i u_i)_{i \in N}$ $\forall u \in \mathbb{R}^N$ and $\alpha.S \equiv \{\alpha.u/u \in S\}$.

Let $\pi_{N\setminus\{i\}}(S)$ be the projection of S over $\mathbb{R}^{N\setminus\{i\}} \times \{0\}$, where $i \in N$.

Example :- Let $u_i(S) = \dfrac{\max_{u \in S} \prod_{j \in N} u_j}{\max_{u \in \pi_{N\setminus\{i\}}} \prod_{j \in N\setminus\{i\}} u_j}$, $\forall i \in N$,

if $S \neq \{0\}$; $\bar{u}_i(S) = 0$ $\forall i \in N$, if $S = \{0\}$. Then $\bar{u}:\Sigma_0 \to \mathbb{R}^N$ is a reference function satisfying anonymity and scale independence. This reference function is derived from the Nash collective utility function.

We require a social choice function φ to satisfy two properties:

(i) Anonymity : For any permutation σ of N and any vector $u \in \mathbb{R}^N_+$, write $\sigma(u)$ for the vector where $\sigma(u)_i = u_{\sigma(i)}$ $\forall i \in N$. Then $\forall S \in \Sigma_0$: $\varphi(\sigma(S)) = \sigma(\varphi(S))$ where $\sigma(S) = \{\sigma(u)/u \in S\}$.

(ii) Unanimity : For all $S \in \Sigma_0$, $\varphi(S)$ is a Pareto-optimal element of S.

The class of proposed solutions we have in mind is defined as:

$\varphi^*(S) = \bar{\lambda}\,\bar{u}(S)$ where $\bar{\lambda} = \max\{\lambda / \lambda\,\bar{u}(S) \in S\}$.

3. Characterization Theorem For The Proposed Solution :- The similarity of the proposed solution with the egalitarian solution of Kalai (1977) and the relative egalitarian solution of Kalai and Smorodinsky (1975) in spirit cannot be missed. Our characterization theorem reveals the similarity in motivation. We need the following additional assumptions:

Scale Independence : For all $\alpha \in \mathbb{R}^N_{++}$, all $S \in \Sigma_0$: $\varphi(\alpha.S) = \alpha.\varphi(S)$ with $\alpha.u = (\alpha_i u_i)_{i \in N}$ $\forall u \in \mathbb{R}^N$ and $\alpha.S = \{\alpha u/u \in S\}$.

Partial Issue Monotonicity : For all $\alpha \in \Sigma_0$ with $\bar{u}_i(S) = \bar{u}_j(S)$ $\forall i$,

$j \in N$ and for all $S' \in \Sigma_0$ with $S' = \sigma(S')\ \forall$ permutations σ on N, ($S' \subset$ => ($\varphi(S) \geq \varphi(S')$).

<u>Theorem</u> :- (i) φ^* $:\Sigma_0 \to \mathbb{R}^N_+$ satisfies scale independence and partial issue monotonicity.

(ii) If $:\Sigma_0 \to \mathbb{R}^N_+$ satisfies scale independence and partial issue monotonicity then $\varphi = \varphi^*$.

<u>Proof</u> :- The proof of (i) is immediate. So we shall prove (ii). By scale independence we can assume $\bar{u}(S)=e$ (the vector with all components equal to 1). Thus, $\varphi^*(S)=\bar{\lambda}.e$ where $\bar{\lambda}$ is as defined above. Without loss of generality assume $\bar{\lambda}>0$ for if $\bar{\lambda}=0$ then $S=\{0\}$ and there is nothing to prove.

By minimal transferability, $\forall i \in N$, $\exists v^i \in S$ with $v^i_i < \bar{\lambda}$ and $v^i_j > \bar{\lambda}\ \forall j \neq i$. Let

$$\alpha = \min_{1 \leq i \neq j \leq n} v^i_j$$

Observe, $\alpha > \bar{\lambda}$. Consider vectors $a^i \in \mathbb{R}^N_+$ with $a^i_i = 0$ and $a^i_j = \alpha\ \forall j \neq i$. By comprehensiveness $a^i \leq v^i \Rightarrow a^i \in S\ \forall i \in N$.

Consider the set T=convex hull $\{0, a^1, \ldots, a^n, \bar{\lambda}e\}$. It is easy to check that $T \in \Sigma_0$, $T \subseteq S$ and $\sigma(T)=T\ \forall \sigma: N \to N$ which are bijective. By anonymity and unanimity, $\varphi(T)=\bar{\lambda}e$. By partial issue monotonicity, $\varphi(S) \geq \bar{\lambda}.e$. By Pareto optimality of $\bar{\lambda}.e$ in S we have $\varphi(S)=\bar{\lambda}.e$.

Q.E.D.

4. Conclusion :- Among the other merits of the above solution the one that stands out foremost is that it is characterized without either Nash's Independence of Irrelevant assumption (a controversial assumption in the literature) or without issue monotonicity (an assumption implying Independence of Irrelevant alternatives in the presence of Pareto optimality). The solution makes use of the 'reference function' in a non-trivial sense and overcomes non-optimality. Thus its importance.

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