Sensitivity Analysis for the Single Row Facility Layout Problem

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Abstract

The single row facility layout problem (SRFLP) is an important combinatorial optimization problem where a given set of facilities have to be arranged in a single row so as to minimize the weighted sum of the distances between all pairs of facilities. Sensitivity analysis for the SRFLP has not been reported in the literature till date. In this paper we present closed form expressions for tolerances of all SRFLP parameters. We also present heuristics to obtain upper bounds on the values of these tolerances. Our computational experiments show that the heuristics obtain exact values of tolerances for small sized instances. For larger sized instances, our heuristics obtain good quality bounds on the values of tolerances for a large fraction of the problem parameters. We also present a tightening procedure to improve on the upper bounds generated by our heuristics.

Keywords: Single row facility layout problem; Sensitivity analysis; Tolerances; Upper bounds; Heuristics;

1 Introduction

The Single Row Facility Layout Problem (SRFLP) is formally defined as follows.

Given: A set $F = \{1, 2, \ldots, n\}$ of n facilities, $n > 2$; length $l_j \geq 0$ of facility j $\forall j \in F$; transmission intensity $c_{ij} \geq 0$ of facility pair (i, j), $\forall (i, j) \in F \times F$.

Objective: To obtain a permutation $S = (s_1, s_2, \ldots, s_n)$ of facilities in F that minimizes the cost

$$z(S) = \sum_{i=1}^{n-1} \sum_{j=i+1}^{n} c_{s_i s_j} d_{s_i, s_j},$$

where $d_{s_i s_j} = (l_{s_i} + l_{s_j})/2 + \sum_{k=i+1}^{j-1} l_{s_k}$.

We denote a SRFLP instance using the notation $\mathcal{I} = (F, L, C)$ where F is a given set of facilities, $L = (l_i)$ is a n-dimensional vector of lengths, and $C = [c_{ij}]$ is a $n \times n$ matrix of transmission intensities. Individual values of facility lengths and transmission intensities are the parameters in a SRFLP instance.

The SRFLP has numerous practical applications (see, e.g., Simmons 1969) and is known to be NP-Hard (Beghin-Picavet and Hansen 1982). Methods of solving the problem to optimality (see, e.g., Anjos and Yen 2009, Hungerländer and Rendl 2011, Amaral and Letchford 2012, for recent studies), as well as methods to obtain good quality solutions for the problem within reasonable times (see, e.g., Samarghandi and Eshghi 2010, Datta et al. 2011, for recent studies) have been studied extensively in the literature.

The study of postoptimality analysis (see, e.g., Nauss 1979, Gal and Greenberg 1997) is an important aspect in optimization problems. It deals with the stability of a given optimal solution to changes in instance data. Specifically, given a problem instance and an optimal solution to the instance, postoptimality analysis obtains bounds within which one or more problem parameters can vary without compromizing the optimality of the given optimal solution. Such an analysis is useful for hard optimization problems, and has received much attention in the literature (see, e.g., Greenberg 1998, and references therein). In practice, hard combinatorial optimization problems are often used to model problem situations in which problem parameters change frequently. Recomputing optimal solutions every time a parameter changes is not advisable, given that the problem is hard. Hence the results from postoptimality analysis helps a decision maker to ascertain whether or not a particular set of changes in data requires reoptimization.

One approach of postoptimality analysis is sensitivity analysis. In sensitivity analysis, the value of exactly one problem parameter is allowed to vary, and the analysis computes the bounds within which the parameter value can vary so that an optimal solution to the instance remains optimal after the change in the value of that problem parameter. To the best of our knowledge, sensitivity analysis for the SRFLP has not been studied in the literature. In this paper we present sensitivity analysis results for the SRFLP. We use the tolerance approach to perform sensitivity analysis. In this approach a SRFLP instance and an optimal solution to the instance are given, and upper and lower tolerances for each SRFLP parameter are defined as follows.

Definition 1 (Upper Tolerance) *Given a SRFLP instance $\mathcal{I} = (F, L, C)$ and an optimal solution S^* to $\mathcal{I}$, the upper tolerance β_p of parameter p is the maximum amount by which the value v_p of p can be increased such that S^* remains an optimal solution to the instance obtained by increasing v_p by that amount in $\mathcal{I}$.*

Definition 2 (Lower Tolerance) *Given a SRFLP instance $\mathcal{I} = (F, L, C)$ and an optimal solution S^* to $\mathcal{I}$, the lower tolerance α_p of parameter p is the maximum amount by which the value v_p of p can be decreased while maintaining its non-negativity such that S^* remains an optimal solution to the instance obtained by decreasing v_p by that amount in $\mathcal{I}$.*

If the value v_p of a problem parameter p lies in the interval $(v_p - \alpha_p, v_p + \beta_p)$ where α_p and β_p are lower and upper tolerances respectively, then the given optimal solution to the instance remains optimal for the instance.

The remainder of the paper is organized as follows. In Section 2 we present expressions for upper and lower tolerances for each transmission intensity parameter and each facility length parameter. In Section 3 we present four heuristics to compute upper bounds to the tolerance expressions presented in Section 2. In Section 4 we present computational experiments that demonstrate the quality of bounds output by the heuristic. In Section 5 we present an approach to tighten the bounds that are obtained using the heuristics in Section 3. We summarize the paper in Section 6 and point out directions for future research in this area.

2 Tolerances for SRFLP parameters

In this section we provide expressions for upper and lower tolerances for all parameters in a SRFLP instance. Subsection 2.1 deals with the transmission intensity parameters and Subsection 2.2 deals with the facility length parameters.

2.1 Tolerances for transmission intensity parameters

Consider a SRFLP instance with n facilities, in which the length of facility i is l_i and the transmission intensity between facilities i and j is c_{ij}. Let $\mathcal{S}$ be the set of all solutions to the instance and consider a solution $S \in \mathcal{S}$. Let $z(S)$ be the cost of solution S. Then

$$z(S) = \sum_{i=1}^{n-1} \sum_{j=i+1}^{n} c_{ij} d_{ij}, \tag{1}$$

where d_{ij} is the distance between the centroids of facilities i and j. Now as Simmons (1969) points out,

$$d_{ij} = (l_i + l_j)/2 + b_{ij}^S,$$

where b_{ij}^S is the sum of the lengths of the facilities between i and j in S. So,

$$z(S) = \sum_{i=1}^{n-1} \sum_{j=i+1}^{n} c_{ij}(l_i + l_j)/2 + \sum_{i=1}^{n-1} \sum_{j=i+1}^{n} c_{ij} b_{ij}^S. \tag{2}$$

Note that the first term in the right hand side of Equation (2) is identical for all solutions to the instance. Thus minimizing

$$O(S) = \sum_{i=1}^{n-1} \sum_{j=i+1}^{n} c_{ij} b_{ij}^S$$

to obtain an optimal solution to the SRFLP is an equivalent problem.

Now consider two facilities p and q in S with p located to the left of q. Then the objective function $z(S)$ can be written as

$$z(S) = c_{pq}(l_p + l_q)/2 + c_{pq} b_{pq}^S + \sum_{i=1}^{n-1} \sum_{\substack{j=i+1 \\ (i,j)\neq(p,q)}}^{n} c_{ij}(l_i + l_j)/2 + \sum_{i=1}^{n-1} \sum_{\substack{j=i+1 \\ (i,j)\neq(p,q)}}^{n} c_{ij} b_{ij}^S. \tag{3}$$

If the value of c_{pq} increases (decreases) the rate r_{pq}^S at which the cost of the solution S increases (respectively, decreases) is

$$r_{pq}^S = (l_p + l_q)/2 + b_{pq}^S. \tag{4}$$

Equation (4) leads us to the following lemma.

Lemma 3 *If the transmission intensity c_{pq} between two facilities p and q increases (decreases) in a SRFLP instance, then the rate of increase (respectively, decrease) in the cost of a solution S in which p and q are adjacent is the lowest among all solutions to the instance.*

Proof: The rate of increase (decrease) in such cases is given by Equation (4). Since $b_{pq}^S \geq 0$ for all instances, and $b_{pq}^S = 0$ only when p and q are adjacent, the lemma holds. ■ The next set of results present the expressions for the upper and lower tolerances of transmission intensity parameters.

Theorem 4 *Consider a SRFLP instance and an optimal solution S^* to the instance. If facilities p and q are not adjacent in S^* then the upper tolerance β_{pq} for c_{pq} is*

$$\beta_{pq} = \min_{S \in \mathcal{S}\setminus\{S^*\},\, b_{pq}^{S^*} > b_{pq}^S} \left\{ \frac{z(S) - z(S^*)}{b_{pq}^{S^*} - b_{pq}^S} \right\}.$$

Proof: Since $z(S) \geq z(S^*)$ we restrict ourselves to solutions $S \in \mathcal{S} \setminus S^*$ such that $b_{pq}^{S^*} > b_{pq}^{S}$. Since S^* is optimal, $z(S) \geq z(S^*)$. If $c_{pq} \uparrow \delta$, $\delta > 0$, small; then $z(S) \uparrow r_{pq}^{S}\delta$, and $z(S^*) \uparrow r_{pq}^{S^*}\delta$. If $\delta > (z(S) - z(S^*))/(r_{pq}^{S^*} - r_{pq}^{S})$ then after the change $z(S) < z(S^*)$.

Thus solution S^* ceases to be optimal after an increase in the value of c_{pq} by δ. This happens for the first time when the increase in the value of c_{pq} is more than $\min_{S \in \mathcal{S}\setminus\{S^*\}}\{(z(S) - z(S^*))/(r_{pq}^{S^*} - r_{pq}^{S})\}$. Using equation (4) the theorem follows. ■

Remark 1 *Theorem 4 has the following implications:*

1. *If facilities p and q are adjacent in an optimal solution then the upper tolerance for c_{pq} is ∞.*
2. *If there are multiple optimal solutions, then the upper tolerance limit for c_{pq} is 0 in all optimal solutions but the ones in which b_{pq} is the minimum.*

Theorem 5 *Consider a SRFLP instance and an optimal solution S^* to the instance. If facilities p and q occupy the two extreme positions in S^*, then the lower tolerance for c_{pq} is the value of c_{pq}*

Proof: Consider any solution $S \in \mathcal{S}\setminus\{S^*\}$. Since S^* is optimal, $z(S) \geq z(S^*)$. Since $b_{pq}^{S^*} = \sum_{\substack{i=1 \\ i \neq p,q}}^{n} l_i$ in S^*, $b_{pq}^{S^*} \geq b_{pq}^{S}$ for every $S \in \mathcal{S}$, and so $r_{pq}^{S^*} \geq r_{pq}^{S}$ for every $S \in \mathcal{S}$. If $c_{pq} \downarrow \delta$, $\delta > 0$, small; then $z(S^*) \downarrow r_{pq}^{S^*}\delta$ and $z(S) \downarrow r_{pq}^{S}\delta$. Clearly after such a change, $z(S)$ cannot be less than $z(S^*)$ as $\delta > 0$ and $r_{pq}^{S^*} \geq r_{pq}^{S}$. The theorem follows. ■

Theorem 6 *Consider a SRFLP instance and an optimal solution S^* to the instance. If facilities p and q do not occupy the two extreme positions in S^*, then the lower tolerance α_{pq} for c_{pq} is*

$$\alpha_{pq} = \min\left\{c_{pq}, \min_{S \in \mathcal{S}\setminus\{S^*\},\, b_{pq}^{S^*} < b_{pq}^{S}} \left\{\frac{z(S) - z(S^*)}{b_{pq}^{S} - b_{pq}^{S^*}}\right\}\right\}.$$

Proof: Since $z(S) \geq z(S^*)$ we restrict ourselves to solutions $S \in \mathcal{S} \setminus S^*$ such that $b_{pq}^{S^*} < b_{pq}^{S}$. Since S^* is optimal, $z(S) \geq z(S^*)$. If $c_{pq} \downarrow \delta$, $\delta > 0$, small; then $z(S) \downarrow r_{pq}^{S}\delta$, and $z(S^*) \downarrow r_{pq}^{S^*}\delta$. If $\delta > (z(S) - z(S^*))/(r_{pq}^{S} - r_{pq}^{S^*})$ then after the change $z(S) < z(S^*)$. Thus solution S^* ceases to be optimal after an decrease in the value of c_{pq} by δ. This happens for the first time when the decrease in the value of c_{pq} is more than $\min_{S \in \mathcal{S}\setminus\{S^*\}}\{(z(S) - z(S^*))/(r_{pq}^{S} - r_{pq}^{S^*})\}$. The theorem then follows using equation (4) and the fact that transmission intensities are non-negative. ■

Remark 2 *Theorem 6 implies that if there are multiple optimal solutions, then the lower tolerance for c_{pq} is 0 in the ones in which b_{pq} is the minimum among all optimal solutions.*

2.2 Tolerances for facility length parameters

Consider a solution S to a SRFLP instance $\mathcal{I} = (F, L, C)$. Given a particular facility p, the set of facilities can be partitioned into three sets; set L consisting of all facilities to the left of p, set $\{p\}$, and set R consisting of all facilities to the right of p. The cost of S can be written as

$$\begin{aligned}
z(S) &= \sum_{\substack{[i,j] \\ i,j \in L\cup\{p\}\cup R}} c_{ij}\Big((l_i + l_j)/2 + b_{ij}\Big) \\
&= \sum_{\substack{[i,j] \\ i,j \in L\cup R}} c_{ij}(l_i + l_j)/2 + \sum_{\substack{[i,j] \\ i,j \in L}} c_{ij}b_{ij} + \sum_{\substack{[i,j] \\ i,j \in R}} c_{ij}b_{ij} + \\
&\quad \sum_{j \in L\cup R} c_{pj}(l_p + l_j)/2 + \sum_{j \in L\cup R} c_{pj}b_{pj} + \sum_{i\in L}\sum_{j \in R} c_{ij}b_{ij}. \qquad (5)
\end{aligned}$$

Now suppose that the value of l_p increases (decreases) by δ. This change causes no change in b_{ij} values if i and j both belong to either L or R. If one of i or j is p, then the value of b_{ij} increases (respectively, decreases) by $\delta/2$. If one of i and j belongs to L and the other to R then the value of b_{ij} increases (respectively, decreases) by δ. This means that a change in the value of l_p does not affect the first three terms of Equation (5) but affects its last three terms. Hence the rate of change of the value of $z(S)$ due to a change in the value of l_p is given by

$$r_p^S = \sum_{j \in L\cup R} c_{pj}/2 + \sum_{i\in L}\sum_{j\in R} c_{ij}. \qquad (6)$$

If p is at an extreme end of a solution S, then $r_p^S = \sum_{j\neq p} c_{pj}/2$ and there does not exist a solution S' such that $r_p^{S'} < r_p^S$.

In the next few theorems we present expressions for upper and lower tolerances for facility length parameters.

Theorem 7 *Consider a SRFLP instance and an optimal solution S^* to the instance. If facility p does not occupy an extreme position in S^*, then the upper tolerance β_p of l_p is*

$$\beta_p = \min_{S\in\mathcal{S}\setminus\{S^*\},\, r_p^{S^*} > r_p^S} \left\{ \frac{z(S) - z(S^*)}{r_p^{S^*} - r_p^S} \right\}.$$

Proof: Since $z(S) \geq z(S^*)$ we restrict ourselves to solutions $S \in \mathcal{S} \setminus S^*$ such that $r_p^{S^*} > r_p^S$. Since S^* is optimal, $z(S) \geq z(S^*)$. If $l_p \uparrow \delta$, $\delta > 0$, small; then $z(S) \uparrow r_p^S\delta$, and $z(S^*) \uparrow r_p^{S^*}\delta$. If $\delta > (z(S) - z(S^*))/(r_p^{S^*} - r_p^S)$ then after the change $z(S) < z(S^*)$. Thus solution S^* ceases to be optimal after an increase in the value of l_p by δ. This happens when the increase in the value of l_p exceeds $\min_{S\in\mathcal{S}\setminus\{S^*\}}\{(z(S) - z(S^*))/(r_p^{S^*} - r_p^S)\}$. The theorem follows. ■

Remark 3 *Since the rate of change in costs of solutions with respect to a change in the length of a particular facility is independent of the length of the facility, the result in Theorem 7 is unaffected by the existence of multiple optimal solutions.*

Theorem 8 *Consider a SRFLP instance and an optimal solution S^* to the instance. If facility p occupies an extreme position in S^*, then the upper tolerance β_p of l_p is ∞.*

Proof: Assume to the contrary, that the upper tolerance of l_p is $u < \infty$. Let S be a solution whose cost is lower than that of S^* when $l_p \uparrow (u + \varepsilon)$, $\varepsilon > 0$, small. Without loss of generality, assume that facility p occurs at the right-most position in S^*.

Since S^* is optimal, $z(S) \geq z(S^*)$. As a consequence of Equation (6) the costs of all solutions with p at an extreme end increases at the same rate when the value of l_p increases. So, facility p

cannot be present at an extreme end of S. Now if we partition the facilities into three sets, set L of facilities to the left of p in S, set $\{p\}$, and set R of facilities to the right of p in S. Then

$$\begin{aligned} r_p^S - r_p^{S^*} &= \Big(\sum_{j \in L \cup R} c_{pj}/2 + \sum_{i \in L} \sum_{j \in R} c_{ij} \Big) - \Big(\sum_{j \neq p} c_{pj}/2 \Big) \\ &= \sum_{i \in L} \sum_{j \in R} c_{ij} \geq 0. \end{aligned}$$

So the cost of the postulated solution S is not lower than the cost of S^*, and when the value of l_p increases, the rate of increase in the cost of S is at least the same as that of of S^*. Hence the cost of S^* cannot be higher than the cost of S after any finite increase in the value of l_p. This contradicts our assumption and the theorem follows. ■

Theorem 9 *Consider a SRFLP instance and an optimal solution S^* to the instance. The lower tolerance α_p for l_p is*

$$\alpha_p = \min \left\{ l_p, \min_{S \in \mathcal{S} \setminus \{S^*\},\, r_p^{S^*} < r_p^S} \left\{ \frac{z(S) - z(S^*)}{r_p^S - r_p^{S^*}} \right\} \right\}.$$

Proof: Since $z(S) \geq z(S^*)$ we restrict ourselves to solutions $S \in \mathcal{S} \setminus S^*$ such that $r_p^{S^*} < r_p^S$. Since S^* is optimal, $z(S) \geq z(S^*)$. If $l_p \downarrow \delta$, $\delta > 0$, small; then $z(S) \downarrow r_p^S \delta$, and $z(S^*) \downarrow r_p^{S^*} \delta$. If $\delta > (z(S) - z(S^*))/(r_p^S - r_p^{S^*})$ then after the change $z(S) < z(S^*)$. Thus solution S^* ceases to be optimal after a decrease in the value of l_p by δ. This happens when the decrease in the value of l_p is at least $\min_{S \in \mathcal{S} \setminus \{S^*\}} \{(z(S) - z(S^*))/(r_p^S - r_p^{S^*})\}$. The theorem now follows using the fact that facility lengths are non-negative. ■

3 Heuristics for computing bounds on tolerances

The literature on sensitivity analysis for combinatorial optimization problems point to the result that the computational complexity of sensitivity analysis for a combinatorial optimization problem is exactly as hard as the problem itself (see, e.g., Wagelmans 1990, Ramaswamy 1994). This leads us to believe that the computational complexity of sensitivity analysis of the SRFLP is NP-Hard. Given this, it is unlikely that we will have algorithms that can obtain tolerances for all transmission intensity and facility length parameters in time polynomial in the size of the SRFLP instance. Hence, in this section we propose heuristics that provide upper bounds on the tolerances for these problem parameters.

We present four heuristics to compute upper and lower tolerances for parameters, and all of them are along similar lines. Hence we only provide details about the heuristic to compute upper tolerances for transmission intensity parameters. The reader will observe that heuristics to compute lower tolerances for transmission intensity parameters and upper and lower tolerances for facility length parameters are obtained by making minor modifications in the heuristic described below.

Recall from Theorem 4 that the upper tolerance for transmission intensity c_{pq} is given by the expression

$$\beta_{pq} = \min_{S \in \mathcal{S} \setminus \{S^*\},\, b_{pq}^{S^*} > b_{pq}^S} \left\{ \frac{z(S) - z(S^*)}{b_{pq}^{S^*} - b_{pq}^S} \right\}.$$

So in order to compute β_{pq} a solution method needs to search for a solution S^u among all those solutions S in which $b_{pq}^S < b_{pq}^{S^*}$ and the ratio $(z(S) - z(S^*))/(b_{pq}^{S^*} - b_{pq}^S)$ is the minimum. The number of such solutions is potentially exponential in the size of the problem. For example, if facilities p

and q are at the two extreme positions in S^*, then the search for S^u is over the space of all solutions to the instance which is exponential in the size of the problem. In order to reduce computational expenses, our heuristic generates a list $\mathcal{L}$ of k candidate solutions, where k is specified by the user, and then chooses the solution S^a in the list with the lowest $(z(S^a) - z(S^*))/(b_{pq}^{S^*} - b_{pq}^{S^a})$ value as an approximation for S^u.

Our heuristic generates candidate solutions in $\mathcal{L}$ by repeating the following process until k solutions are generated in $\mathcal{L}$. It randomly generates a permutation of the facilities in F and subjects it to local optimization using an insertion neighborhood. (See Kothari and Ghosh 2012, for a description of the insertion neighborhood for the SRFLP.) It examines the solutions that it obtains at each iteration of the local optimization procedure and creates a short list consisting of solutions S in which $b_{pq}^S < b_{pq}^{S^*}$. It finally choses a solution S' from this short list for which the value of $(z(S') - z(S^*))/(b_{pq}^{S^*} - b_{pq}^{S'})$ is the least and adds it to $\mathcal{L}$. In case the short list is empty, the process does not add any solution to $\mathcal{L}$. Once $\mathcal{L}$ is populated the heuristic chooses a solution $S^a \in \mathcal{L}$ with the lowest $(z(S) - z(S^*))/(b_{pq}^{S^*} - b_{pq}^S)$ ratio and computes an upper bound B_{pq} on the upper tolerance β_{pq} as

$$B_{pq} = \left\{ \frac{z(S^a) - z(S^*)}{b_{pq}^{S^*} - b_{pq}^{S^a}} \right\}.$$

A pseudocode for this algorithm is given below.

ALGORITHM COMPUTE-$\mathbf{B_{pq}}$

Input: A SRFLP instance (n, L, C); an optimal solution S^*; facilities p and q; k.
Output: An upper bound B_{pq} to β_{pq}.

Code

1. begin
2. set $\mathcal{L} \leftarrow \emptyset$, $i \leftarrow 0$;
3. while $i < k$ do begin
4. set $S \leftarrow$ random permutation of the n facilities in the given instance;
5. set $tmp \leftarrow \{S\}$;
6. perform local search on S using the insertion neighborhood structure, and add the best neighbor obtained at the end of each iteration of the local search to tmp;
7. remove all solutions S from tmp in which $b_{pq}^S \geq b_{pq}^{S^*}$;
8. if $tmp \neq \emptyset$ then begin
9. set $S' \leftarrow \arg\min\{(z(S) - z(S^*))/(b_{pq}^{S^*} - b_{pq}^S) | S \in tmp\}$;
10. set $\mathcal{L} \leftarrow \mathcal{L} \cup \{S'\}$, $i \leftarrow i + 1$;
11. end;
12. end;
13. set $S^a \leftarrow \arg\min\{(z(S) - z(S^*))/(b_{pq}^{S^*} - b_{pq}^S) | S \in \mathcal{L}\}$;
14. set $B_{pq} = (z(S^a) - z(S^*))/(b_{pq}^{S^*} - b_{pq}^{S^a})$;
15. end.

The above pseudocode can be easily modified to compute the lower tolerances for the trasmission intensity parameters and upper and lower tolerances for the facility length parameters. We call these heuristics COMPUTE-A_{pq}, COMPUTE-B_p, and COMPUTE-A_p respectively. We present the pseudocodes for these heuristics below without further explanation.

ALGORITHM COMPUTE-$\mathbf{A_{pq}}$

Input: A SRFLP instance (n, L, C); an optimal solution S^*; facilities p and q; k.

Output: An upper bound A_{pq} to α_{pq}.

Code

```
 1. begin
 2.     set $\mathcal{L} \leftarrow \emptyset$, $i \leftarrow 0$;
 3.     while $i < k$ do begin
 4.         set $S \leftarrow$ random permutation of the $n$ facilities in the given instance;
 5.         set $tmp \leftarrow \{S\}$;
 6.         perform local search on $S$ using the insertion neighborhood structure,
            and add the best neighbor obtained at the end of each iteration of
            the local search to $tmp$;
 7.         remove all solutions $S$ from $tmp$ in which $b^S_{pq} \leq b^{S^*}_{pq}$;
 8.         if $tmp \neq \emptyset$ then begin
 9.             set $S' \leftarrow \arg\min\{(z(S) - z(S^*))/(b^S_{pq} - b^{S^*}_{pq})|S \in tmp\}$;
10.             set $\mathcal{L} \leftarrow \mathcal{L} \cup \{S'\}$, $i \leftarrow i + 1$;
11.         end;
12.     end;
13.     set $S^a \leftarrow \arg\min\{(z(S) - z(S^*))/(b^S_{pq} - b^{S^*}_{pq})|S \in \mathcal{L}\}$;
14.     set $A_{pq} = \min\left\{c_{pq}, ((z(S) - z(S^*))/(b^S_{pq} - b^{S^*}_{pq})\right\}$;
15. end.
```

ALGORITHM COMPUTE-$\mathbf{B_p}$

Input: A SRFLP instance (n, L, C); an optimal solution S^*; facilities p and q; k.
Output: An upper bound B_p to β_p.

Code

```
 1. begin
 2.     set $\mathcal{L} \leftarrow \emptyset$, $i \leftarrow 0$;
 3.     while $i < k$ do begin
 4.         set $S \leftarrow$ random permutation of the $n$ facilities in the given instance;
 5.         set $tmp \leftarrow \{S\}$;
 6.         perform local search on $S$ using the insertion neighborhood structure,
            and add the best neighbor obtained at the end of each iteration of
            the local search to $tmp$;
 7.         remove all solutions $S$ from $tmp$ in which $r^S_p \geq r^{S^*}_p$;
                                              /* $r_p$ defined in Equation (6) */
 8.         if $tmp \neq \emptyset$ then begin
 9.             set $S' \leftarrow \arg\min\{(z(S) - z(S^*))/(r^{S^*}_p - r^S_p)|S \in tmp\}$;
10.             set $\mathcal{L} \leftarrow \mathcal{L} \cup \{S'\}$, $i \leftarrow i + 1$;
11.         end;
12.     end;
13.     set $S^a \leftarrow \arg\min\{(z(S) - z(S^*))/(r^{S^*}_p - r^S_p)|S \in \mathcal{L}\}$;
14.     set $B_p = (z(S^a) - z(S^*))/(r^{S^*}_p - r^{S^a}_p)$;
15. end.
```

ALGORITHM COMPUTE-$\mathbf{A_p}$

Input: A SRFLP instance (n, L, C); an optimal solution S^*; facilities p and q; k.
Output: An upper bound A_p to α_p.

Code

```
 1. begin
 2.     set $\mathcal{L} \leftarrow \emptyset$, $i \leftarrow 0$;
```

while $i < k$ do begin
    set $S \leftarrow$ random permutation of the n facilities in the given instance;
    set $tmp \leftarrow \{S\}$;
    perform local search on S using the insertion neighborhood structure, and add the best neighbor obtained at the end of each iteration of the local search to tmp;
    remove all solutions S from tmp in which $r_p^S \leq r_p^{S^*}$;
        /* r_p *defined in Equation (6)* */
    if $tmp \neq \emptyset$ then begin
        set $S' \leftarrow \arg\min\{(z(S) - z(S^*))/(r_p^S - r_p^{S^*}) | S \in tmp\}$;
        set $\mathcal{L} \leftarrow \mathcal{L} \cup \{S'\}$, $i \leftarrow i + 1$;
    end;
end;
set $S^a \leftarrow \arg\min\{(z(S) - z(S^*))/(r_p^S - r_p^{S^*}) | S \in \mathcal{L}\}$;
set $A_p = \min\left\{l_p, (z(S^a) - z(S^*))/(r_p^{S^a} - b_p^{S^*})\right\}$;
end.

Note that in the search for S^a in all the algorithms above, we use a random search procedure instead of more conventional population based search procedures like genetic algorithms, scatter search, or particle swarm optimization. This is because in most population based search procedures, the aim of the algorithm is to make the population converge to a small number of solutions with "good" objective values. In our algorithms, this is precisely opposite of what we want to achieve, since if the population converges, the number of unique solutions in $\mathcal{L}$ becomes too low for us to obtain good quality upper bounds on the tolerances.

4 Computational experiments

We now describe our experiments to test the quality of bounds obtained by the four heuristics described above. Our heuristics described in Section 3 require a SRFLP instance and an optimal solution to the instance as an input. The method that we use for computing the quality of bounds differ based on the size of the instance being considered. We set the size k of the list $\mathcal{L}$ of candidate solutions at 1000.

4.1 Quality of bounds obtained for small sized instances

Consider a parameter p having a value of v_p in a given problem instance with a given optimal solution S^*. Also suppose that we obtain an upper bound on the upper (or lower) tolerance of p as B_p (respectively, A_p). We first change the problem data, so that the value of the parameter is $v_p + B_p - \epsilon$ (respectively, $v_p - A_p + \epsilon$) with $\epsilon > 0$ and small, say 0.001; and re-solve the problem to optimality using exhaustive enumeration. If the cost of an optimal solution to the changed instance matches the cost of S^* computed using the data from the changed instance, then the bound obtained is indeed the true value of the appropriate tolerance. If this is not so, then we conclude that the bound obtained is a strict overestimate of the true value of the appropriate tolerance.

Our computations show that all the bounds that we obtained using the four heuristics described in Section 3 are indeed the true values of the optimal tolerances. We report the bounds obtained for these instances below.

Instance P4 of size 4 (Simmons 1969) The optimal solution that we consider for this instance is (2,1,4,3) with cost 638.0. The upper and lower tolerances for the transmission intensity parameters are given below.

B_{pq}	1	2	3
2	∞		
3	0.61	0.39	
4	∞	4	∞

A_{pq}	1	2	3
2	1.27		
3	2.00	3.00	
4	2.54	1.27	0.82

The upper and lower tolerances for the facility length parameters are given below.

	1	2	3	4
B_p	10.40	∞	∞	2.80
A_p	3.50	2.00	2.00	23.00

Instance LW5 of size 5 (Love and Wong 1976) The optimal solution that we consider for this instance is (2,1,5,3,4) with cost 151.0. The upper and lower tolerances for the transmission intensity parameters are given below.

B_{pq}	1	2	3	4
2	∞			
3	1.14	1.00		
4	0.00	0.00	∞	
5	∞	2.00	∞	0.00

A_{pq}	1	2	3	4
2	1.59			
3	0.00	0.00		
4	0.00	2.00	3.82	
5	0.67	0.70	0.00	2.00

The upper and lower tolerances for the facility length parameters are given below.

	1	2	3	4	5
B_p	0.50	∞	0.00	∞	2.33
A_p	1.00	1.00	4.00	0.00	6.20

Instance S8 of size 8 (Simmons 1969) The optimal solution that we consider for this instance is (4,6,8,3,5,1,2,7) with cost 801.0. The upper and lower tolerances for the transmission intensity parameters are given below.

B_{pq}	1	2	3	4	5	6	7
2	∞						
3	1.33	0.50					
4	1.00	0.50	4.14				
5	∞	4.50	∞	3.75			
6	1.00	0.50	2.00	∞	2.00		
7	4.00	∞	0.64	3.04	0.80	2.00	
8	1.00	0.50	∞	2.67	3.75	∞	3.47

A_{pq}	1	2	3	4	5	6	7
2	1.50						
3	4.00	1.00					
4	1.00	2.00	0.00				
5	4.00	3.75	0.80	0.80			
6	3.00	2.00	0.64	2.00	0.00		
7	1.00	0.50	2.00	0.00	3.75	3.00	
8	0.00	2.67	0.64	1.00	0.80	3.50	2.00

The upper and lower tolerances for the facility length parameters are given below.

	1	2	3	4	5	6	7	8
B_p	1.33	4.39	1.40	∞	0.57	8.43	∞	1.14
A_p	0.80	0.33	1.88	3.04	4.50	0.67	2.08	2.00

Instance S8h of size 8 (Heragu and Kusiak 1988) The optimal solution that we consider for this instance is (2,3,6,4,5,1,8,7) with cost 2324.5. The upper and lower tolerances for the transmission intensity parameters are given below.

B_{pq}	1	2	3	4	5	6	7
2	1.11						
3	1.77	∞					
4	0.80	3.00	3.00				
5	∞	1.00	1.00	∞			
6	1.89	4.80	∞	∞	1.00		
7	2.00	2.82	1.11	0.80	1.70	1.50	
8	∞	0.89	0.89	0.80	2.00	1.14	∞

A_{pq}	1	2	3	4	5	6	7
2	2.00						
3	2.00	1.11					
4	2.00	0.80	0.80				
5	1.00	1.70	1.70	3.00			
6	2.00	1.50	1.50	0.80	2.80		
7	1.11	5.00	4.00	3.00	1.00	5.20	
8	1.11	6.00	4.00	3.00	1.00	5.00	0.89

The upper and lower tolerances for the facility length parameters are given below.

	1	2	3	4	5	6	7	8
B_p	0.46	∞	0.67	0.79	0.51	1.24	∞	1.35
A_p	1.15	0.71	0.91	4.00	5.00	0.92	1.21	0.42

Instance S9 of size 9 (Simmons 1969) The optimal solution that we consider for this instance is (2,3,6,9,1,5,7,4,8) with cost 2469.5. The upper and lower tolerances for the transmission intensity parameters are given below.

B_{pq}	1	2	3	4	5	6	7	8
2	1.00							
3	5.67	∞						
4	0.33	1.00	8.00					
5	∞	0.50	0.50	7.25				
6	4.00	1.00	∞	4.00	0.50			
7	0.33	1.00	2.07	∞	∞	2.07		
8	0.33	1.00	6.00	∞	6.00	4.00	6.86	
9	∞	1.00	9.00	4.83	0.50	∞	5.80	4.83

A_{pq}	1	2	3	4	5	6	7	8
2	0.00							
3	0.33	8.00						
4	5.67	0.00	1.12					
5	6.22	2.00	1.12	0.50				
6	0.33	4.00	1.12	0.00	5.00			
7	0.00	4.00	0.00	2.07	0.50	5.73		
8	1.00	4.00	1.12	8.00	0.50	2.00	2.07	
9	0.33	4.83	1.12	6.00	6.00	4.83	5.73	6.00

The upper and lower tolerances for the facility length parameters are given below.

	1	2	3	4	5	6	7	8	9
B_p	0.25	∞	0.39	1.92	7.33	8.17	2.00	∞	5.12
A_p	2.00	0.33	6.30	2.67	0.33	2.00	2.64	1.72	7.11

Instance S9h of size 9 (Simmons 1969) The optimal solution that we consider for this instance is (2,5,9,1,7,3,6,4,8) with cost 4695.5. The upper and lower tolerances for the transmission intensity parameters are given below.

B_{pq}	1	2	3	4	5	6	7	8
2	0.00							
3	2.50	0.00						
4	0.00	0.00	0.20					
5	2.00	∞	0.00	1.28				
6	0.00	0.00	∞	∞	0.17			
7	∞	0.00	∞	1.64	0.00	0.17		
8	0.00	0.00	0.20	∞	0.27	4.00	2.54	
9	∞	0.00	0.00	0.46	∞	0.17	0.80	0.50

A_{pq}	1	2	3	4	5	6	7	8
2	1.50							
3	0.20	0.20						
4	3.44	4.00	0.00					
5	0.00	0.00	0.00	0.00				
6	4.14	8.00	0.00	0.17	0.00			
7	2.43	2.75	0.20	0.00	0.00	0.00		
8	4.14	6.00	0.00	0.23	0.00	0.17	0.00	
9	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00

The upper and lower tolerances for the facility length parameters are given below.

	1	2	3	4	5	6	7	8	9
B_p	1.61	∞	0.00	0.23	0.00	2.00	0.00	∞	0.00
A_p	0.00	0.00	2.57	0.73	0.44	0.06	6.00	0.67	1.06

Instance S10 of size 10 (Heragu and Kusiak 1988) The optimal solution that we consider for this instance is (8,6,2,4,10,5,7,1,3,9) with cost 2781.5. The upper and lower tolerances for the transmission intensity parameters are given below.

B_{pq}	1	2	3	4	5	6	7	8	9
2	4.75								
3	∞	4.75							
4	7.22	∞	5.00						
5	11.00	1.14	5.00	1.14					
6	4.00	∞	4.00	4.00	1.14				
7	∞	4.75	5.00	7.22	∞	4.00			
8	7.07	2.00	5.00	6.33	1.14	∞	9.73		
9	3.33	3.67	∞	3.67	3.67	3.67	3.67	3.67	
10	4.00	4.75	4.00	∞	∞	4.00	4.00	6.00	3.67

A_{pq}	1	2	3	4	5	6	7	8	9
2	0.00								
3	5.50	2.00							
4	3.33	2.00	0.00						
5	1.00	2.00	1.14	6.33					
6	3.33	4.75	2.00	7.22	0.00				
7	3.33	2.00	5.00	0.00	1.14	4.00			
8	3.33	4.75	0.00	5.00	7.00	4.00	0.00		
9	7.00	0.00	5.00	2.00	1.14	3.00	2.00	1.00	
10	0.00	2.00	4.00	4.00	10.15	4.00	6.00	4.00	0.00

The upper and lower tolerances for the facility length parameters are given below.

	1	2	3	4	5	6	7	8	9	10
B_p	2.31	0.67	1.94	4.19	9.57	3.27	3.38	∞	∞	4.00
A_p	6.00	3.00	2.50	4.00	0.73	1.09	7.67	2.93	1.07	7.00

Instance LW11 of size 11 (Love and Wong 1976) The optimal solution that we consider for this instance is (6,4,9,1,5,7,2,8,3,10,11) with cost 6933.5. The upper and lower tolerances for the transmission intensity parameters are given below.

B_{pq}	1	2	3	4	5	6	7	8	9	10
2	4.40									
3	1.18	1.43								
4	3.56	3.56	1.18							
5	∞	5.71	1.18	3.56						
6	22.00	5.71	1.18	∞	7.33					
7	4.40	∞	1.43	3.56	∞	15.59				
8	4.40	∞	∞	3.56	10.80	9.44	13.33			
9	∞	5.71	1.18	∞	7.33	6.40	11.00	9.44		
10	4.40	6.67	∞	3.56	22.12	7.91	6.67	3.33	11.00	
11	4.40	6.67	14.33	2.70	11.78	2.12	6.67	3.33	4.15	∞

A_{pq}	1	2	3	4	5	6	7	8	9	10
2	2.00									
3	0.00	0.00								
4	4.40	4.00	0.00							
5	14.11	6.67	7.33	15.00						
6	0.00	5.00	0.00	3.56	12.00					
7	6.67	12.33	6.00	6.67	6.67	2.00				
8	3.33	3.33	2.00	3.33	3.33	2.00	3.33			
9	4.40	6.40	6.40	11.00	6.00	8.00	6.40	3.33		
10	7.00	5.71	1.18	1.00	7.00	1.00	4.00	9.37	1.00	
11	3.00	0.00	1.18	7.00	3.00	0.00	6.00	5.77	6.40	2.00

The upper and lower tolerances for the facility length parameters are given below.

	1	2	3	4	5	6	7	8	9	10	11
B_p	4.66	1.75	2.57	7.80	1.60	∞	2.98	0.71	0.86	2.35	∞
A_p	1.57	3.00	0.28	0.44	5.00	2.25	7.00	4.15	2.00	3.07	3.43

4.2 Quality of bounds obtained for larger sized instances

For larger sized problem instances, the use of exhaustive enumeration to compute optimal solutions is prohibitively expensive. Studies in the literature (see, e.g., Hungerländer and Rendl 2011, Amaral and Letchford 2012) show that even advanced algorithms require significant amounts of time to obtain optimal solutions to these instances. Given that in order to test all bounds on a problem of size n we need to obtain optimal solutions to at least $n(n+1)$ instances of the same size. we use scatter search to generate good approximations to optimal solutions. We choose scatter search since our initial experiments show that scatter search is able to obtain optimal solutions to all benchmark instances with sizes up to 30.

Since the output of scatter search is not guaranteed to be optimal, our check for the quality of bounds is slightly different for these instances. Suppose we are given an instance and an optimal solution to the instance. Consider a parameter p having a value of v_p in a given problem instance with a given optimal solution S^*. Also suppose that we obtain an upper bound on the upper (or lower) tolerance of p as B_p (respectively, A_p). We first change the problem data, so that the value of the parameter is $v_p + B_p - \epsilon$ (respectively, $v_p - A_p + \epsilon$), with $\epsilon > 0$ and small, say 0.001; and re-solve the problem using scatter search. Suppose the solution returned by scatter search is S^s. (Remember that this may not be an optimal solution.) If the costs of S^s and S^* are identical for the changed instance, then we assume that the bound that we obtained is the true value of the appropriate tolerance. If this is not the case, and if the cost of S^* is higher than the cost of S^s with the changed data, then we conclude that the bound that we have is not the true value of the appropriate tolerance. If the cost of S^* is lower, then we infer that scatter search has not obtained an optimal solution to the instance with the changed data and we cannot conclude whether or not the bound obtained by our heuristics matched the actual values of the appropriate tolerances.

We report the results of our experiments on SRFLP benchmark instances in Tables 1 and 2. The structures of both these tables are similar. In the first two columns we report the name of the instance and its size. The third column reports the number of bounds for tolerance values that were checked. For a problem instance of size n, there are $n(n-1)$ bounds on tolerances for transmission intensities (see Table 1) and $2n$ bounds on tolerances for facility lengths (see Table 2). The last three columns in the tables report the number of these bounds which were found to be optimal (i.e., where the costs of S^* and S^s were identical), suboptimal (i.e., where the cost of S^* was higher than the cost of S^s), and where we could not conclude anything about the optimality of the bound obtained (i.e., where the cost of S^* was lower than the cost of S^s). This option is conceivable since scatter search is a heuristic which does not guarantee to output optimal solutions.

From the tables we see that when computing the tolerances for transmission intensity parameters, our heuristics were able to output optimal values in approximately 85% of the cases, while they could output optimal values of tolerances for facility length parameters in approximately 46% of the cases. We also observed that optimal values of tolerances were output more often when computing lower tolerances than when computing upper tolerances. This may be because of the fact that the lower tolerance values were truncated to the value of the parameter when they were too high to satisfy the condition of non-negativity of problem parameters. We also see that our conjecture that scatter search produces high quality solutions for these problems is empirically validated.

5 Tightening of the bounds on tolerances

Our computational experiments in Section 4 show that the bounds that we obtain through our heuristics in Section 3 are not the actual tolerance values in many cases, especially for larger instances. In this section therefore, we present a generic heuristic that iteratively tightens the bounds output by the heuristics in Section 3. The heuristic that we propose can be combined with each of the four heuristics to produce tighter bounds on the appropriate tolerance. It depends on the

Table 1: Quality of bounds for tolerances on transmission intensities for larger instances

instance	size	Nr. of tolerances	Quality of bounds obtained[a] optimal	suboptimal	inconclusive
P15[b]	15	210	208	2	0
H20[c]	20	380	342	38	0
H30[c]	30	870	668	202	0
N-25-01[d]	25	600	555	45	0
N-25-02[d]	25	600	546	54	0
N-25-03[d]	25	600	497	103	0
N-25-04[d]	25	600	547	53	0
N-25-05[d]	25	600	543	57	0
N-30-01[d]	30	870	690	179	0
N-30-02[d]	30	870	726	144	0
N-30-03[d]	30	870	766	104	0
N-30-04[d]	30	870	658	209	3
N-30-05[d]	30	870	682	188	0

a: based on scatter search results
Source b: Heragu and Kusiak (1991)
c: Heragu and Kusiak (1988)
d: Anjos and Vannelli (2008)

Table 2: Quality of bounds for tolerances on facility lengths for larger instances

instance	size	Nr. of tolerances	Quality of bounds obtained[a] optimal	suboptimal	inconclusive
P15[b]	15	30	26	4	0
H20[c]	20	40	24	16	0
H30[c]	30	60	22	38	0
N-25-01[d]	25	50	15	35	0
N-25-02[d]	25	50	30	20	0
N-25-03[d]	25	50	28	22	0
N-25-04[d]	25	50	31	19	0
N-25-05[d]	25	50	23	27	0
N-30-01[d]	30	60	11	49	0
N-30-02[d]	30	60	25	35	0
N-30-03[d]	30	60	31	29	0
N-30-04[d]	30	60	21	37	2
N-30-05[d]	30	60	25	35	0

a: based on scatter search results
Source b: Heragu and Kusiak (1991)
c: Heragu and Kusiak (1988)
d: Anjos and Vannelli (2008)

empirical fact that scatter search produces solutions that are very close to optimal for small and medium sized SRFLP instances. For illustration purposes, we describe this heuristic for tightening the upper bound on the upper tolerance for a transmission intensity parameter. It is easy to extend the same argument for all the other bounds obtained by the other heuristics in Section 3.

Suppose that we want to tighten the upper bound on the upper tolerance for c_{pq} obtained using COMPUTE-B_{pq}. The upper bound is given as input to the heuristic, along with the SRFLP instance and an optimal solution S^* to the instance. Each iteration of the heuristic starts with an upper bound u on the upper tolerance. Scatter search is used to compute a good quality solution S^s to the instance obtained by increasing the value of c_{pq} by u. Let the costs of S^* and S^s with the original instance data be z_o^* and z_o^s; and with the value of c_{pq} reduced by u be z_n^* and z_n^s. Denoting $z_o^s - z_o^*$ by Δ_o and $z_n^* - z_n^s$ by Δ_n, the value of u is then revised to $u\Delta_o/(\Delta_o + \Delta_n)$ if $\Delta_n > 0$ and left unchanged otherwise. Iterations stop when there is no revision in the value of u, and the value of u after the last iteration is output by the heuristic.

The rationale behind the iterative process is the following. Suppose that the bound u that we have at the beginning of an iteration is not the actual tolerance for the parameter. Then when the corresponding parameter is increased by u, the original optimal solution S^* ceases to be optimal and a different solution becomes optimal, i.e., has lower cost than S^* for the changed instance. We use scatter search to obtain a near optimal solution to the instance with the changed data, and use this solution S^s as an approximation for the optimal solution to the changed instance. The iteration then finds the the value of u for which the cost of S^* matches the cost of S^s with the changed data. Beyond this, S^* definitely is suboptimal, and hence the new value of u is saved as an improved bound. This process fails when the cost of S^s for the changed instance is not lower than the cost of S^* for the changed instance, i.e., when scatter search cannot find a good quality solution to the changed instance, or when the value of the bound is indeed the exact tolerance value. In these cases the bound is not changed.

We cannot test the quality of bounds obtained after the tightening, since that would require an exact and fast algorithm to obtain optimal solutions to SRFLP instances. It does not make sense to test the quality of bounds using scatter search output, since scatter search was used to obtain the tightened bounds in the first place. We however checked to see whether the tightening process described in this section improved the bounds in the 1378 cases where the bounds obtained by the heuristics in Section 3 were clearly suboptimal (see Tables 1 and 2). We noticed that the tightening process improved 841 of these bounds. This leads us to conclude that the tightening process described in this section is indeed useful to obtain good quality bounds.

6 Summary of contribution

In this paper we perform sensitivity analysis for the single row facility location problem (SRFLP). The problem is an important one in facility layout, and is frequently used to model real world situations in which facilities have to be arranged in a single line with an objective of minimizing total inter-facility communication costs. We use the tolerance approach to perform sensitivity analysis and obtain upper and lower tolerances with the property that if the value of a problem parameter increases (or decreases) by a value not exceeding the upper (respectively, lower) tolerance, then the optimality of a given optimal solution is not compromised.

In Section 2 we present closed form expressions for the values of upper and lower tolerances for each problem parameter. Most of these results require information about another solution to the problem instance having particular properties. Since the search for such solutions does not seem to be possible in time polynomial to the size of the instance being considered, in Section 3 we present heuristics that output bounds to the tolerance values. We then perform computational experiments to test the quality of these bounds in Section 4. The first part of the section shows that the bounds are in fact optimal for small problem sizes. The second part of the section demonstrates that the bounds obtained by the heuristics are not always optimal for medium sized instances. In Section 5 we propose a method to tighten the bounds output by the heuristics described in Section 3. We observe that the tightening procedure improves more than 60% of the bounds that were found to be suboptimal in our experiments in Section 4.

The work presented here can be advanced in several directions. For example, one could use specialized algorithms to obtain optimal solutions to SRFLP instances of medium size, and hence test the quality of the bounds obtained after the tightening process described in Section 5. One could also perform stability analysis for the problem, in which more than one problem parameter could vary simultaneously.

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