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A NOTE ON (S,s) INVENTORY POLICY

By

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A NOTE ON (S,s) INVENTORY POLICY

N. RAVICHANDRAN*

ABSTRACT

This article obtains the stationary distribution of the stock level of a Continuous review (S,s) Inventory policy with the following characteristics. The demand rate is a constant λ and unit quantity is demanded whenever there is a demand. When the stock level reduces to s an order for (S - s) units is placed which materialises after a random time ; the lead time is assumed to have distribution of PH - type. Further, it is assumed that the demands that arise when the stock level is zero are lost.

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1. Introduction:

This article studies the stochastic process induced by a continuous review (S, s) inventory model with poisson demand and non-instantaneous replenishment. More specifically, the stationary distribution of the stochastic process $\{X(t), t \geq 0\}$, where $X(t)$ represents the stock level at any time t of the (S, s) ordering inventory policy is obtained in an easily computable form for a class of distributions governing the lead time. It is assumed that unit quantity is demanded at constant rate λ, and the demands that arise when the stock level is zero are lost. The inventory system is continuously reviewed and an order for (S - s) units is placed immediately when the stock level reduces to s. The order thus placed materialises after a random time interval whose distribution is taken to be of PH - type [2] representing a fairly wide class of distributions including the generalised Erlangian and mixtures of exponentials. The inventory level at any time is the physical inventory available at that time. We also stipulate that $S - s > s$; as a consequence of which at any time at most only one order can be pending. Also, because of the lost sales case assumption, the above stipulation on the recorder quantity is essential for the dynamics of the system. References [3, 4, 5] discuss similar (S, s) ordering inventory models. However the present article is different from these in the sense that it incorporates noninstantaneous lead time.

2. Phase type distributions:

A (continuous) probability distribution of phase type (PH - distribution) on $(0, \infty)$ is one which can be obtained as that of the time till absorption in a finite state continuous time Markov Chain with one absorbing state and all other states are transient. To be specific, consider a Markov Chain on 1, 2, ... m+1 with initial probability vector $(\underline{\alpha}, 0)$ where $\underline{\alpha} \in R^m$ is such that $\underline{\alpha} \geq 0, \underline{\alpha}\,\underline{e} = 1$ and infinitesimal generator

$$Q^* = \begin{bmatrix} T & \underline{T}^0 \\ \underline{0} & 0 \end{bmatrix}$$

Where T is a non-singular m x m matrix with $T_{ii} < 0$ and $T_{ij} \geq$ for $i \neq j$ and $T\,\underline{e} + \underline{T}^0 = \underline{0}$ where $\underline{e} = (1, 1, \ldots.. 1)' \in R^m$. For such a Markov Chain, the time till absorption in (m+1) has cdf.

$F(x) = 1 - \underline{\alpha} \exp(T x)\,\underline{e}, \; x \geq 0.$ The pair (α, T) is called a representation of the PH- distribution F(.). We say that the PH-distribution is in phase j $(1 \leq j \leq m)$ if the underlying Markov Chain is in state j. For several important and useful properties of PH-type distributions we refer the reader to [2].

3. Problem formulation:

We consider a continuous review (S, s) ordering inventory policy with poisson demand and PH- lead time. Let X(t) denote the state of the process representing the on hand inventory at any time t. For any fixed time t, X(t) is a random variable taking any of the integral values from 0 to S. Then $\{X(t), t \geq 0\}$ is a discrete valued continuous time parameter stochastic process. The points of discontinuity of X(t) are either time epochs at which there is a demand or those epochs at which a replenishment occur. In the former case the stock level drops down by unity and in the later case the inventory level increases by a quantity equal to S − s. When the stock level is $\leq$ s, we observe that an order has been placed at the previous time epoch, when the stock level just dropped down to s and is pending. In view of the assumption that the lead time is a random variable, we cannot predict the future behaviour of the process X(t) given that X(t) $\leq$ s without some additional information. The additional information required is the time elapsed since the last reorder. However, because of the assumption that the lead time is a random variable of phase type it is not necessary to specify the time elapsed since the last reorder as a continuous variable. On the other hand, because of the exponential sojourn time in each phase of the phase type distribution, it is sufficient to specify the additional information by the phase of the random variable representing the lead time. We supplement this information for X(t) whenever X(t) $\leq$ s by means of an ordered pair (s', j) $0 \leq s' \leq s$ and $1 \leq j \leq m$. Denote by vector $\underline{i}$ $(0 \leq i \leq s)$ the set of states $\{(i,j) ; 1 \leq j \leq m\}$. Hence the state space of the stochastic process X(t), which is easily seen to be a continuous time parameter Markov Chain is

$$S' = \{S, S-1, S-2, \ldots\ldots s+1, \underline{s}, \underline{s-1} \ldots \underline{1}, \underline{0}\}$$

The generator of $\{X(t), t \geq 0\}$ is given by a square matrix of order and its elements denoted by q_{ij} are given by

$q_{ij} = T$	$i = o$ and $j = o$
$= \alpha \, T^{o}_{\alpha}$	$i = o$ and $j = s$
$= T - \lambda I$	$1 < i < s$ and $j = i$
$= \lambda I$	$1 \leq i \leq s$ and $j = i-1$
$= T^{o}$	$1 \leq \; \leq s$ and $j = i+s$
$= -\lambda$	$i \geq s+1$ and $j = i$
$= \lambda\alpha$	$i = s+1$ and $j = i-1$
$= \lambda$	$i > s+1$ and $j = i-1$
$= 0$	Otherwise

Remarks:

Whenever $i \leq s$ the elements in the matrix are to be interpred as blocks. They are no more single entries. For example

$$q_{s+1 \quad s+1} = -\lambda$$

$$q_{s-1,\, s} = \underline{0} \; (m \times m)$$

$$q_{s+1,\, s-2} = \lambda I \; (m \times m)$$

$$q_{s+1 \; s} = \lambda\alpha \; (1 \times m)$$

$$q_{s \quad s-1} = (T - \lambda I) \; (m \times m)$$

Stationary distribution of $X(t)$

We are interested in the limiting distribution of $X(t)$ if it exists. For the inventory model under consideration it clear that the stationary distribution exists because of the irreducibility of the Markov Chain representing the fluctuations in the stock level. Let $\bar{\Lambda}$ denote the vector representing the stationary distribution of $\{X(t), t \geq 0\}$ and let $\bar{\Lambda} = [\underline{X}_0, \underline{X}_1, \underline{X}_2, \ldots \underline{X}_s, X_{s+1} \ldots\ldots X_S]$

where $X_i = \lim_{t \to \infty} pr\{X(t) = i\}$, $s \leq i \leq S$

and $\underline{X}_i = (X_{i1}, X_{i2}, \ldots X_{im})$

with $X_{ij} = \lim_{t \to \infty} pr\{X(t) = (i,j)\}$ $0 \leq i \leq s;\ 1 \leq j \leq m.$

The vector $\bar{\Lambda}$ is determined from the equations $\bar{\Lambda}\ Q = 0$ and $\sum_{i=0}^{S} \bar{\Lambda}_i = 1$ which are equivalent to the following set of equations.

(1) $\underline{X}_0\ T + \underline{X}_1\ \lambda I = \underline{0}$

(2) $\underline{X}_i\ (T - \lambda I) + \underline{X}_{i+1}\ \lambda I = \underline{0}$ $1 \leq i \leq s-1$

(3) $\underline{X}_s\ (T - \lambda I) + X_{s+1}\ \lambda \underline{a} = \underline{0}$

(4) $-\lambda X_i + \lambda X_{i+1} = 0$ $s < i < S-s$

(5) $\underline{X}_0\ \underline{T}^0 - \lambda X_{S-s} + \lambda X_{S-s+1} = 0$

(6) $\underline{X}_i\ T^0 - \lambda X_{S-s+i} + \lambda X_{S-s+i+1} = 0$ $1 < < s-1$

(7) $\underline{X}_s\ T^0 - \lambda X_S = 0$

(8) $\sum_{i=0}^{s} \underline{X}_i e + \sum_{i=s+1}^{S} X_i = 1$

When $S-s = s$ equations (3) - (5) in the above are replaced by the following equation

$$\underline{X}_0\ T^0 \underline{a} + \underline{X}_s\ (T - \lambda I) + X_{s+1}\ \lambda \underline{a} = \underline{0}$$

It remains to solve the set of equations (1) - (8) to obt the stationary distribution of $\{X(t), t \geq 0\}$

From equation (4) we easily have

(9) $X_i = C$ $\qquad s+1 \leq i \leq S-s$

where $C = X_{s+1}$

From equation (1) we obtain

(10) $\underline{X}_1 = \underline{X}_0 TK$

where $K = -(\lambda I)^{-1}$

using equation (10) after some simplifications equation (2) resul

(11) $\underline{X}_i = \underline{X}_0 TK \; M^{i-1}$ $\qquad 2 \leq i \leq s$

Where $M = (T - \lambda I) K$

From equations (5) - (7) we obtain

(12) $X_{S-s+1} = C - \lambda^{-1} \underline{X}_0 T^0$

(13) $X_{S-s+j} = C - \lambda^{-1} \underline{X}_0 T^0 - \underline{X}_0 U_j T^0$ $\qquad 2 \leq j \leq s$

Where $U_j = \lambda^{-1} TK \left(\sum_{i=1}^{j-1} M^{i-1}\right), J \geq 2$

and finally from equations (4) and (3)

(14) $\underline{X}_s = -C \lambda \underline{\alpha} (T - \lambda I)^{-1}$

using equation (11) in (14) results

(15) $\underline{X}_0 = - C \lambda \underline{\alpha} N^{-1}$

Where $N = TKM^{s-1} (T - \lambda I)$

Now applying the normality condition we obtain

(16) $X_{s+1} = \frac{1}{P}$

where $P = S - s - \lambda \times \underline{\alpha}\, N^{-1} [\{ I + \sum_{i=1}^{s} TKM^{i-1} \} e$

$$- \sum_{j=2}^{s} U_j T^o - s\lambda^{-1} T^o]$$

Thus using equations (9) – (16) the stationary distribution of the process is completely determined. We also observe that the stationary distribution is easily computable from the above relations since in the process we only need the inverse and powers of the matrix $(T - \lambda I)$, which are possible from elementary considerations.

Special case:

We consider the case of exponential lead time with parameter μ In this case the various symbols introduced earlier are given by

$$T = -\mu ; T^o = \mu : \alpha = (1) : e = (1),\ I = \qquad (1)$$

$$K = -\frac{1}{\lambda}\ M = \frac{\lambda+\mu}{\lambda}$$

$$U_j = -\frac{1}{\lambda} \left\{ 1 - \left(\frac{\lambda+\mu}{\lambda}\right)^{k-1} \right\} \qquad k \geq 2$$

$$N^{-1} = -\frac{1}{\mu} \left(\frac{\lambda}{\lambda+\mu}\right)^s$$

and the stationary probabilities are given by

$$\underline{X}_0 = C - \frac{\lambda}{\mu} \left(\frac{\lambda}{\lambda+\mu}\right)^s$$

$$\underline{X}_i = C \left(\frac{\lambda}{\lambda+\mu}\right)^{s-i+1} \qquad 1 \leq i \leq s$$

$$X_i = C \qquad s+1 \leq i \leq S-s$$

$$X_{S-s+1} = C - C \left(\frac{\lambda}{\lambda+\mu}\right)^s$$

$$X_{S-s+j} = C - C \left(\frac{\lambda}{\lambda+\mu}\right)^{s-j+1} \qquad 2 \leq j \leq s$$

and $$C = \frac{1}{(S+s) + \frac{\lambda}{\mu} \left(\frac{\lambda}{\lambda+\mu}\right)^s} \qquad 2\ \ j\ \ s$$

We further simplify to the special case corresponding to s=o, in which case the constant

$$C = \frac{1}{S + \frac{\lambda}{\mu}}$$

and $\underline{X}_o = C\frac{\lambda}{\mu}$

$X_i = C \qquad\qquad i \leq i \leq S$

Next, we further specialise to the case of instantaneous replenishment, which is obtained by letting $\mu \to \infty$. Thus we get,

$$X_i = \frac{1}{S-s} \text{ for } i = s+1, \ldots\ldots\ldots S$$

a result in agreement with [5].

We now consider the special case S = 6, s = 3. The generator of the Markov Chain representing the stock level is given by

$$\begin{array}{c} \\ \underline{0} \\ \underline{1} \\ \underline{2} \\ \underline{3} \\ 4 \\ 5 \\ 6 \end{array} \begin{array}{c} \begin{array}{ccccccc} \underline{0} & \underline{1} & \underline{2} & \underline{3} & 4 & 5 & 6 \end{array} \\ \begin{bmatrix} T & 0 & 0 & T^o_{\mu\lambda} & 0 & 0 & 0 \\ \lambda I & T-\lambda I & 0 & 0 & T^o & 0 & 0 \\ 0 & \lambda I & T-\lambda I & 0 & 0 & T^o & 0 \\ 0 & 0 & \lambda I & T-\lambda I & 0 & 0 & T^o \\ 0 & 0 & 0 & \lambda\underline{\alpha} & -\lambda & 0 & 0 \\ 0 & 0 & 0 & 0 & \lambda & -\lambda & 0 \\ 0 & 0 & 0 & 0 & 0 & \lambda & -\lambda \end{bmatrix} \end{array}$$

On solving the set of equations corresponding to (1) – (8) we get

(16) $\underline{X}_1 = \underline{X}_0 \, TK$

(17) $\underline{X}_2 = \underline{X}_0 \, TKM$

(18) $\underline{X}_3 = \underline{X}_0 \, TKM^2$

(19) Where $M = (T - \lambda I)K$ and $K = -(\lambda I)^{-1}$

Further

$$\underline{X}_0 = -X_4 \lambda \underline{\alpha} \, L^{-1}$$

Where $L = T^0 \underline{\alpha} + TKM^2 (T - \lambda I)$

(20) $X_5 = \lambda^{-1} \underline{X}_3 \, T^0$

(21) $X_6 = \lambda^{-1} (\underline{X}_2 + \underline{X}_3) \, T^0$

and

(22) $X_4 = \frac{1}{P}$

Where $P = 1 - \underline{\alpha} \; L^{-1} \, [\, TKM + 2 \, TKM^2 \,] T^0$

$$- \lambda \alpha L^{-1} \, (I + TK + TKM + TKM^2) e$$

Thus the stationary distribution of the stochastic process $\{^3 X(t), \, t \geq 0\}$ for this special case is completely characterised by means of equations (16) – (22).

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