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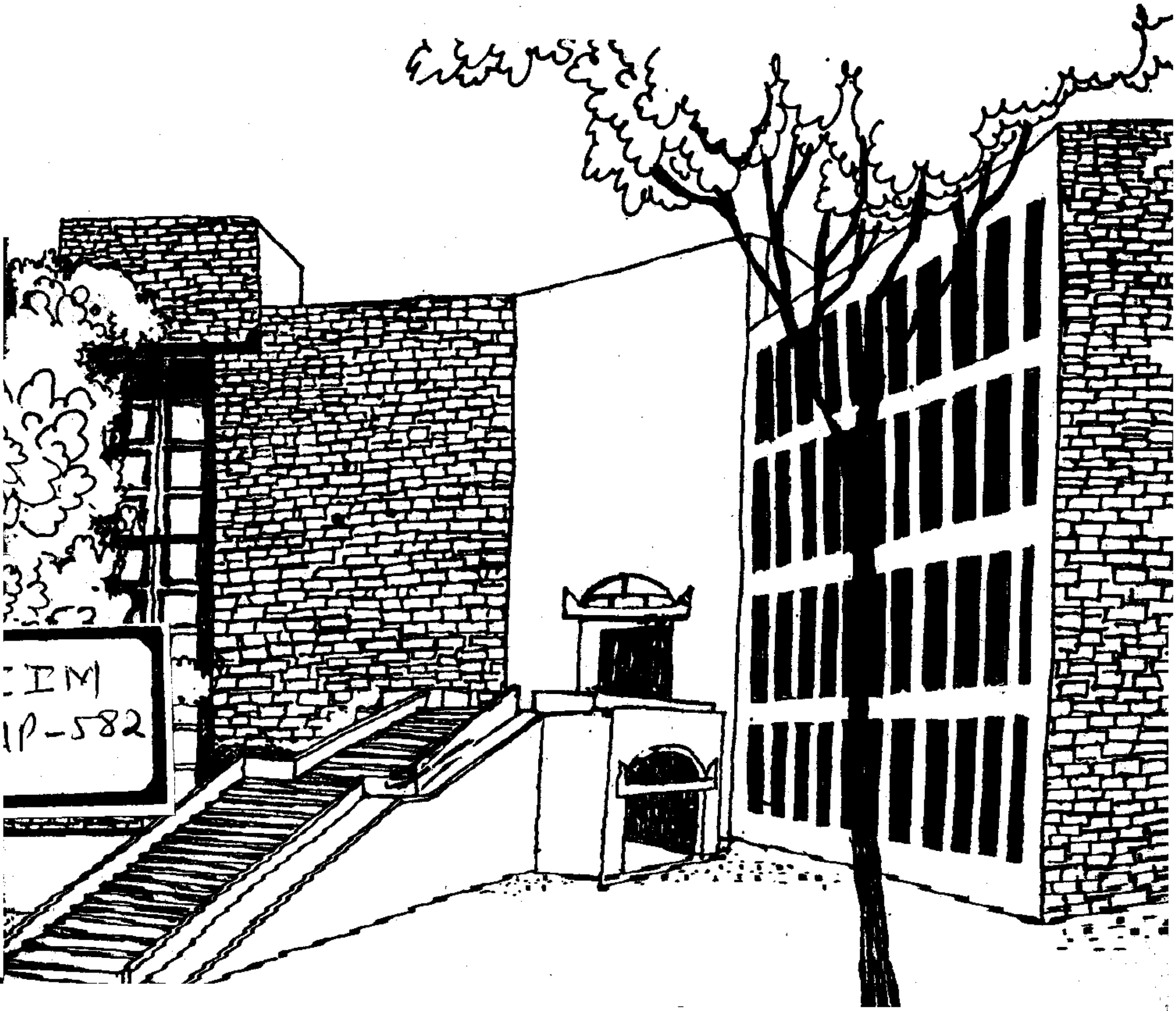

COMPLETE MATCHING IN A TRINOMIAL DOUBLY-CONVEX COMPLETE BIPARTITE GRAPH

By

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COMPLETE MATCHING IN A TRINOMIAL DOUBLY-CONVEX COMPLETE BIPARTITE GRAPH

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Abstract

This paper discusses minimum matching in a trinomial doubly-convex bipartite graph. The graph consists of three categories of arcs forming a doubly-convex structure. The matching involves lexicographical minimization in required order of categories. Due to the special structure of the problem, certain 'greedy' dures are found to be optimal.

COMPLETE MATCHING IN A TRINOMIAL DOUBLY-CONVEX COMPLETE BIPARTITE GRAPH

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Consider a complete bipartite graph G = (S,T, SXT) such that $|S| = |T| = n$. Let a_i and b_j be the real numbers associated with each $i \in S$ and $j \in T$ respectively. Further assume that nodes in S and T are ordered such that

$$a_1 \leqslant a_2 \leqslant \cdots\cdots \leqslant a_n$$
$$b_1 \leqslant b_2 \leqslant \cdots\cdots \leqslant b_n$$

For a given constant g, the arcs can be classified into three categories as follows.

$$R = \{(i,j) : b_j - a_i < \emptyset\}$$
$$B = \{(i,j) : \emptyset \leqslant b_j - a_i < g\}$$
$$Y = \{(i,j) : b_j - a_i \geqslant g\}$$

From the definition of the graph, following properties are evident.

P1. if $(i,j) \in R$ then $(K,j) \in R$ for all $K > i$

P2. if $(i,j) \in R$ then $(i,K) \in R$ for all $K > j$

P3. if $(i,j) \in Y$ then $(K,j) \in Y$ for all $K > i$

P4. if $(i,j) \in Y$ then $(i,K) \in Y$ for all $K > j$

P5. if $(i,j) \in B$ and $(i,K) \in B$ and $J < K$, then $(i,p) \in B$ for all $j < p < K$

P6. if $(i,j) \in B$ and $(K,j) \in B$, and $i < K$, then $(p,j) \in B$ for all $i < p < K$

The properties P5 and P6 hold for categories R and Y as well.

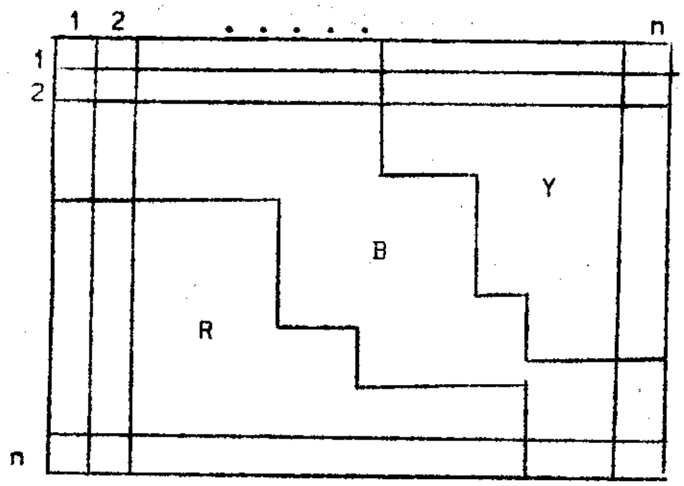
We shall call the graph G with three kinds of arcs and properties P1 through P6 as "Trinomial Doubly-Convex Complete Bipartite Graph". We shall explore complete matching in graph G.

Let $M \subseteq R \cup B \cup Y$ be a complete matching in G such that no two arcs in M are incident on a common node. In general, arcs in M can be partitioned in terms of the three categories $M_R \in R$, $M_B \in B$ and $M_Y \in Y$. There will be situations where we would like to achieve a matching which is lexicographically minimum in the sense that $|M_R|$, $|M_B|$ and $|M_Y|$ are minimised in a hierarchical manner. Incidentally, in view of the complete matching, $|M_Y|$ gets determined as residual effect once we minimise $|M_R|$ followed by $|M_B|$.

The prototype for our problem is vehicle scheduling problem with maintenance constraint. In this problem we have to match n arriving trips with n departing trips at a terminal such that fleet-size requirement is minimised and number of vehicles getting maintenance is maximised. Let S denote set of arrivals, T denote set of departures, a_i and b_j denote corresponding timings, and g be the minimum time required for maintenance. Then M_Y would be the set of matches where vehicle can undergo maintenance in between the two trips. M_B would be the set of matches where a vehicle takes another trip after finishing the arrival trip, but does not have enough idle gap to go through the maintenance. The matches M_R indicate that departure in this category takes place even before the matching arrival, since $b_j - a_i < \emptyset$. In this case, the departure can take place only if additional vehicle is provided for the purpose, and the vehicle available on arrival of i would remain free for rest of the day, possibly catering to the demand for additional vehicle on the next day and so on. The sum of $|M_R|$ over all the terminals would determine the fleet-size required to operate the set of trips.

Algorithm Development

The assignment tableau for the matching problem has a block structure as shown in figure below.



Matching in category R

It is clear that if cells of category R do not figure among the diagonal cells, (1,1) ......(n,n) then, it is possible to reduce $|M_R|$ to zero, since we can simply adopt diagonal matching solution. Minimum number of matches in R is given by following theorem.

Theorem

Minimum number of matches in R is given by

$$|M^*_R| = \max\{\emptyset, j - i+1 : (i,j) \in R\}$$

Proof

Let (i^*,j^*) be the extreme cell satisfying $j^* - i^*+1 = \max\{j - 1+1 : (i,j) \in R\}$

Case 1. $j^* < i^*$

In this case all the cells belonging to R are confined to area below the diagonal. Then the diagonal matching would ensure that $|M^*_R| = \emptyset$.

Case 2. $j^* \geq i^*$

Consider matching in rows i^* to n. There are $n - i^*+1$ such rows. By property P1 and P2, all the cells in the block formed by rows i^* to n and columns 1 to j^*, belong to category R. Therefore, there are at most $n - j^*$ columns available for rows i^* to n for matching in categories other than R. Thus,

$$|M^*_R| \geq (n - i^*+1) - (n - j^*)$$
$$\geq j^* - i^*+1$$

Now we have to show that $|M^*_R|$ can be in fact restricted to $z^* = j^* - i^*+1$

Suppose we restrict z^* matchings in a $z^* \times z^*$ block consisting of columns 1 to z^* and rows $n - z^*+1$ to n. Than in the resulting residual matching problem of $n - z^* \times n - z^*$, all the cells belonging to category R will fall below the new diagonal. Because if we cross-out first z^* columns and last z^* rows, the new column numbers of all the remaining cells would reduce by z^*, while row numbers will remain intact. Therefore, for the extreme cell,

$$(j^* - z^*) - i^*+1 = \emptyset$$

Thus in the residual matching problem, we need not have anymore matches in category R.

The matching in R as indicated in the proof of the theorem not only results in minimum $|M_R|$, but also renders the residual problem capable of being optimised (minimised) in $|M_B|$.

To explore further optimization, we shall use preference operators $\prec$ and $\preceq$ as follows.

Let M* and M be complete matchings in G. Then, $M^* \prec M$ if

$$1. \quad |M_R^*| < |M_R|$$

$$\text{or} \quad 2. \quad |M_R^*| = |M_R| \text{ and } |M_B^*| < |M_B|$$

Likewise, we can extend the above to individual matchings. Thus

$(p,q) \prec (r,s) \prec (t,u)$, where $(p,q) \in R$, $(r,s) \in B$, $(t,u) \in Y$

Using doubly-convex properties,

$(i,j) \preceq (i,k)$ if $j > k$

$(i,j) \preceq (k,j)$ if $i > k$

Theorem

Let $z^* = \max\{j - i+1 : (i,j) \in R\}$ and $z^* > \emptyset$ and M^* is the matching obtainable by restricting matching in category R within a $z^* \times z^*$ block B_0 consisting of columns 1 to z^* and rows $(n - z^*+1)$ to n. To show that if M is any other arbitrary matching, then $M^* \preceq M$.

Proof

From previous theorem, $|M_R| \geqslant z^*$

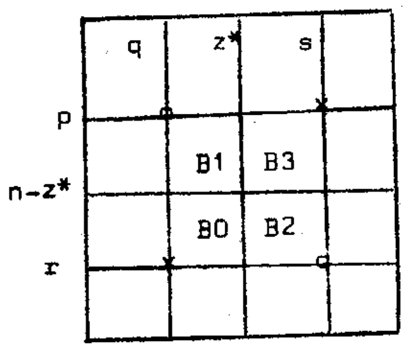
We assume that $|M_R| = z^*$, since otherwise $M^* \prec M$. Though $|M^*_R| = |M_R| = z^*$, M_R differs from M^*_R in that at least one match in it belonging to category R is outside the block B_0. Such a match can be in one of the three blocks, namely

B1 : rows 1 to $n - z^*$ and columns 1 to z^*

B2 : rows $(n - z^*+1)$ to n and columns z^*+1 to n

B3 : rows 1 to $n - z^*$ and columns z^*+1 to n.

Case 1: $(p,q) \in M_R$, $1 \leqslant p \leqslant n-z^*$ and $1 \leqslant q \leqslant z^*$ (Block B1)



Since column q does not contain a match in B_0 there must be a row r within B_0 containing match (r,s) in B_2 outside B_0.

Suppose we modify M to delete (p,q) and (r,s) and add (r,q) and (p,s), such that match in B1 is brought inside B_0,

$$M' = M - (p,q) - (r,s) + (r,q) + (p,s).$$

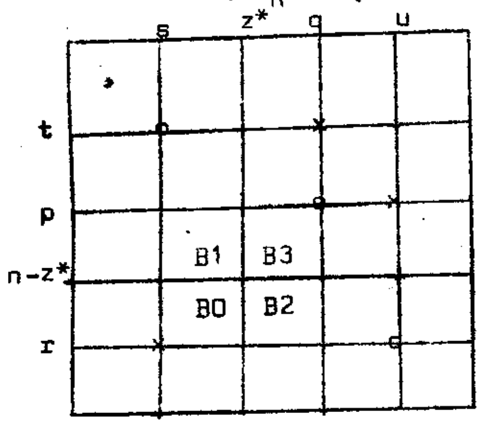
We see that

$$M' \preccurlyeq M \quad \text{since } (p,q) \in R,\ (r,q) \in R \text{ and } (p,s) \preccurlyeq (r,s)$$

Case ii: $(r,s) \in M_R$ $\quad n - z^*+1 \leq r \leq n$ and $z^*+1 \leq s \leq n$ (Block B2)

Similar argument holds if we bring the match in B2 inside B0.

Case iii: $(p,q) \in M_R$ $\quad 1 \leq p \leq n - z^*$, $z^*+1 \leq q \leq n$ (Block B3)



Since block B0 containes less than z^* matches, there must be at least one row r and one column s spanning B0 not having a match within B0. Let $(t,s) \in M$ and $(r,u) \in M$ be the corresponding matches in column s and row r respectively.

Modifying M to replace (p,q), (t,s) and (r,u) by (r,s), (t,q) and (p,u)

$$M' = M - (p,q) - (t,s) - (r,u) + (r,s) + (t,q) + (p,u)$$

We see that

$$M' \preccurlyeq M \text{ since } (p,q) \in R,\ (r,s) \in R,\ (t,q) \preccurlyeq (t,s),\ (p,u) \preccurlyeq (r,u).$$

Matching in Category B

Having minimised $|M^*_R|$ with least deterioration in the residual problem consisting of block B3 of the assignment tableau, we now minimise $|M^*_B|$. The special structure enables us to do it using simple heuristic which is optimal. In the procedure we minimise $|M^*_B|$ by avoiding matching in Category B as far as possible, and when the matching in B is inevitable, we choose the one which results in least deterioration in the residual problem.

Procedure A

```
while matching is not yet complete do
    l ← last row yet to be matched
    if{(l,k) : (l,k) ∈ Y} is not empty
      then
        p ← any cell such that (l,p) ∈ Y
      else
        p ← min{K: (l,k) ∈ B}
    endif
    M ← M U (l,p)
    strike off row l and column p
endwhile
endprocedure
```

Theorem

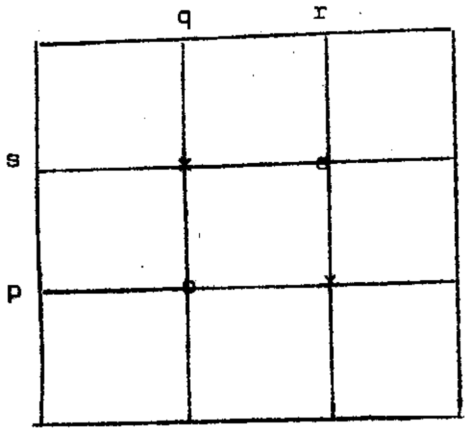
Procedure A results in optimal matching in the residual problem consisting of block B3.

Proof

Let M be an optimal matching. Starting with last row, we check M for its conformity with solution obtainable by the procedure A, striking off corresponding rows and columns if conformity is established. Suppose the solution does not conform at row P. We will show that optimal solution M may be suitably modified to make it conform with that of the procedure A at row P, and the modified solution M' non-conforming at the most at row p' where $p' \leqslant p-1$, will be optimal matching. By induction it would then follow that the procedure is optimal.

Let $(p,q) \in M$ and let (p,r) be the matching indicated by the procedure A. We need to consider two cases, namely $(p,r) \in Y$ and $(p,r) \in B$.

Case 1. $(p,r) \in Y$



Non-conformity with the procedure A suggests that $(p,q) \in B$, since otherwise (p,q) would have conformed with the selection criteria of the procedure. $(p,q) \in B$ and $(p,r) \in Y$ implies that $q < r$ due to block structure properties.

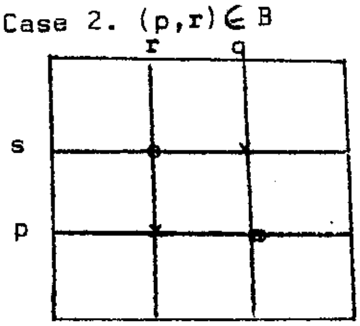
Since matching is to be complete, every row and every column must be matched. Therefore, column r must have a match $(s,r) \in M$ for some $s < p$ since the rows $p+1$, $p+2$, ... n and corresponding matched columns have been already crossed out due to conformity. Suppose we now modify M to make it conform with the procedure A in row p as follows.

$$M' = M - (p,q) - (s,r) + (p,r) + (s,q)$$

We find that

$$M' \prec M \text{ since } (s,r) \in Y,\ (p,r) \in Y \text{ and } (s,q) \preceq (p,q).$$

Case 2. $(p,r) \in B$



Non-conformity with the procedure A suggests that $(p,q) \in B$ and $r < q$, since otherwise (p,q) would have conformed with the selection criteria of procedure A.

As before, $(s,r) \in M$ for some $s < p$.

Modifying M to make it conform with procedure A,

$$M' = M - (p,q) - (s,r) + (p,r) + (s,q)$$

We find that

$$M' \preccurlyeq M \text{ since } (p,q) \in B,\ (p,r) \in B \text{ and } (s,q) \preccurlyeq (s,r).$$

Thus, in both the cases M' is an optimal matching non-conforming with the procedure A at the most at row p', where $p' \leqslant p-1$. Successive modification will eventually leave us with optimal matching fully conforming with the procedure A.

Having explored the lexicographically minimum matching for the hierarchical order (R,B,Y), we shall now explore it for the hierarchical orders (R,Y,B) and (Y,R,B). Prototype problem of assembly of two parts within specified tolerance, relating to these orders was reported by Glover (1967), where he proposed a greedy procedure to maximise the number of assemblies within specified tolerance, corresponding to matches in B. In lexicographically minimum matching version of the problem we extend it to the issue of salvaging of the parts matched outside the specified tolerance. Depending upon the relative cost of salvaging matches in R vis-a-vis those in Y, we need to pursue one of the two hierarchical orders, namely, (R,Y,B) or (Y,R,B). The question whether lexicographically minimum matchings in these orders also give maximum cardinality matching in B, will be answered in affirmative following a greedy procedure to solve the problem. In fact, the procedure seeks to maximise matches in B, leaving a residual problem capable of being optimised in either of the hierarchical orders.

Maximum matching in Category B

We follow a simple greedy procedure which 'grabs-the-nearest-B' for every row starting from row one. The procedure may be formally stated as follows:

```
procedure B
  for i = 1 to n  do
    if {(i,j) : (i,j) ∈ B} is not empty
      then
        p ← min {j: (i,j) ∈ B}
        M ← M U (i,p)
        strike off row i and column p
    endif
  endfor
endprocedure
```

The residual problem leftover by procedure B can then be solved by method developed for matching in category R if hierarchical order (R,Y,B) is desired. The same method can also be adopted for the hierarchical order (Y,R,B), where we compute $z^* = \max\{i-j+1 : (i,j) \in Y, \emptyset\}$ and restrict matching in Y to a block formed by rows 1 to z^* and volumns $n-z^*+1$ to 1.

Optimality of procedure B is proved in the following

Theorem

The procedure B achieves maximum cardinality matching in Category B and renders the residual problem capable of being optimised such that overall matching is lexicographically minimum for either of the hierarchical orders (R,Y,B) or (Y,R,B).

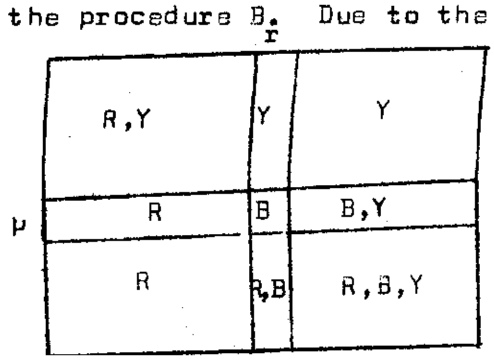
Proof:

We follow a similar proof technique as in previous theorem, where we assume an optimal solution and modify it to conform to our solution procedure.

Let M be an optimal matching. At this stage we are not concerned about the nature of the optimality, namely, maximum cardinality matching in category B, lexicographically minimum matching for (R,Y,B) or for (Y,R,B). We shall analyse the impact of procedure B on each of these cases.

Starting with the first row, we check M for its conformity with the solution obtainable by the procedure B, striking off corresponding rows and columns if conformity is established. Suppose the solution does not conform at row p. We will show that optimal solution M may be suitably modified to make it conform with that of the procedure B at row p, and the modified solution M' non-conforming at the most at row p' where $p' \geq p+1$ will be optimal matching. Continuing the process for all rows would establish the proof.

Let $(p,q) \in M$ and let (p,r) be the matching indicated by the procedure B. Due to the properties of doubly-convex structure and the conditions implied by procedure B, we can make following observations about the cells around (p,r).



1. $(p,r) \in B$
2. $(i,j) \in Y$ for $i \leq 1 < p$ and $r \leq j \leq n$
3. $(i,j) \in R$ for $i \leq p \leq n$ and $1 \leq j < r$
4. $(i,j) \in R \cup Y$ for $i \leq 1 < p$ and $1 \leq j < r$
5. $(i,j) \in R \cup B$ for $p < i \leq n$
6. $(p,j) \in B \cup Y$ for $r < j \leq n$
7. $(i,j) \in R \cup B \cup Y$ for $p < i \leq n$ and $r < j \leq n$

Since matching is to be complete, every row and every column must be matched. Therefore, column r must have a match $(s,r) \in M$. Suppose we now modify M to make it conform with the procedure B in row p as follows.

$$M' = M - (p,q) - (s,r) + (p,r) + (s,q)$$

Let $\delta_R = |M_R| - |M'_R|$, $\delta_B = |M_B| - |M'_B|$, and $\delta_Y = |M_Y| - |M'_Y|$

Depending on relative position of s and q with respect to p and r, the cells participating in the modification of M could assume various values as enumerated in Table 1. A key aspect of this table is that there is no row in which δ_R and δ_Y differ in sign.

Table 1

Position of s and q	(p,q)	(s,r)	(p,r)	(s,q)	δ_R	δ_B	δ_Y
(s < p)&(q < r)	R	Y	B	R	0	+1	–1
	R	Y	B	Y	–1	+1	0
(s < p)&(q > r)	B	Y	B	Y	0	0	0
	Y	Y	B	Y	0	+1	–1
(s > p)&(q < r)	R	R	B	R	–1	+1	0
	R	B	B	R	0	0	0
(s > p)&(q > r)	B	R	B	R	0	0	0
	Y	R	B	R	0	+1	–1
	B	R	B	B	–1	+1	0
	Y	R	B	B	–1	+2	–1
	B	B	B	B	0	0	0
	Y	B	B	B	0	+1	–1
	Y	R	B	Y	–1	+1	0
	Y	B	B	Y	0	0	0

It is clear from the Table 1 that modification of M forcing it to conform with a solution obtainable by the procedure B leads to one of the following.

1. Number of matchings in each of the categories remain unchanged
2. Number of matchings in category B increase with corresponding decrease in matchings in category R and/or category Y.

Therefore,

i) if we assume M to be optimal with respect to the maximum cardinality matching in category B, M' is also optimal, if not better.

ii) if we assume M to be optimal with respect to the lexicographical minimum matching for order (R,Y,B), so would be M'.

iii) if we assume M to be optimal with respect to the lexicographical minimum matching for order (Y,R,B), so would be M'.

Hence the proof.

Conclusion:

The special structure of the trinomial doubly-convex complete bipartite graph enables us to achieve lexicographically minimum matching using simple greedy procedures, that are optimal. The procedures work on heuristic idea that if matchings in a certain categories are inevitable then choose them in such a manner as to cause minimum deterioration to the other categories in the residual problem.

References

F. Glover, "Maximum Matching in a Convex Bipartite Graph" Naval Research Logistic Quarterly, 14(1967), 313-316.