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ON THE NUMBER OF NON-NEGATIVE INTEGRAL SOLUTIONS TO THE KNAPSACK PROBLEM

by

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Title of the report ....On the Number of Non-Negative Integral Solutions to the Knapsack Problem

Name of the Author .... M. Raghavachari and Y.P. Sabharwal ....

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ABSTRACT (within 250 words)

This paper develops expressions for the exact number of solutions to the well known Knapsack problem. These formulae are compared with the bounds given by other researchers in this problem. A computer programme has also been developed to find the number of solutions.

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Date 25.7.1974.....

M. Raghavachari

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Summary: The well-known Knapsack problem is an integer programming problem given by

$$\text{maximize } c_1x_1 + c_2x_2 + \dots + c_m x_n$$

$$\text{subject to } a_1 x_1 + a_2 x_2 + \dots + a_m x_m \leq b \qquad (1)$$

$$x_j \geq 0 \text{ and integer}$$

Here a_i's are positive integers.

M.W. Padberg [4] and A.G. Beged Dov [1] found upper and lower bounds on the number of solutions to the above Knapsack constraint. In this paper we point out that it is possible to calculate the exact number of solutions. Exact expressions for the number of solutions are developed. For a few examples the bounds are compared with the exact solutions.

1. By adding a slack variable x_{m+1} to (1) we can transform the inequality constraint to an equality constraint. Thus, we are interested in finding the number of solutions to the Knapsack equation.

$$a_1x_1 + a_2x_2 + \dots + a_n x_n = b; \; x_i \geq 0 \text{ and integer} \qquad (2)$$

where $a_1, a_2 .. a_n$, b are any given positive integers.

L.E. Dickson [2, p.156] reports that G. Mignosi gave a method for finding the exact number of solutions to (2). With C(i), i=1,2,.. for the number of solutions to (2) with b replaced by i. Let S(j) denote the sum of those of $a_1, a_2 \dots a_n$ which are divisors of j i.e.

$$S(l) = \sum_{j=1}^{n} a_j \delta_{jl}$$

where

$$\delta_{jl} = 1 \text{ if } a_j \text{ is a divisor of } l$$
$$= 0 \text{ otherwise.}$$

Then Mignosi gave the recursion formula

$$S(1)\, C(b-1) + S(2)\, C(b-2) + \dots + S(b)\, C(0) = bC(b) \qquad (3)$$

where C(0) = 1. By using the recursion formula (3) we can compute C(b) for any integer b=1,2,....

Example: Suppose that the Knapsack equation is given by

$$5x_1 + 2x_2 + 4x_3 + 3x_4 + 4x_5 + x_6 = 11$$

$$x_i \geq 0 \text{ and integer; } i=1,2,..6$$

One can easily verify that

S(1) =1; S(2) =3; S(3)=4; S(4) = 11

S(5) =6; S(6) =6; S(7)=1; S(8) = 11

S(9) =4; S(10)=8 and S(11) = 1

Applying (3) we find recursively

$C(1)=1$; $C(2)=2$; $C(3)=3$; $C(4)=6$;

$C(5)=8$; $C(6)=12$; $C(7)=16$; $C(8)=24$

$C(9)=31$; $C(10)=42$; $C(11)=53$

Since the work of Mignosi is not available to us and may not be available to many we give an independent proof of the recursion formula.

Proof of Mignosi's formula: Define the generating function:

$$K(y) = \sum_{b=0}^{\infty} C(b)\, y^b \tag{4}$$

so that

$bC(b)$ is the coefficient of y^b in $yK'(y)$ where prime denotes derivative $C(b)$, however, is the coefficient of y^b in

$$\prod_{j=1}^{n} \left(1-y^{a_j}\right)^{-1}$$

Thus

$$yK'(y) = K(y) \sum_{j=1}^{n} a_j \; y^{a_j}\left(1 - y^{a_j}\right)^{-1}$$

Since

$$y^{a_j}\left(1-y^{a_j}\right)^{-1} = \sum_{l=1}^{\infty} \delta_{jl}\, y^l .$$

We see therefore from (4) that

$$K(y)\, y^{a_j} \left(1-y^{a_j}\right)^{-1} = \sum_{b=1}^{\infty} \left\{ \sum_{k=0}^{b-1} C(K))\, \delta_{j(b-k)} \right\} y^b$$

Hence collecting the coefficients of y^b we have

$$bC(b) = \sum_{j=1}^{n} a_j \sum_{k=0}^{b-1} C(k)\, \delta_{j(b-k)}$$

$$= \sum_{k=0}^{b-1} C(k) \sum_{j=1}^{n} a_j\, \delta_{j(b-k)}$$

$$= \sum_{k=0}^{b-1} C(k)\, S(b-k)$$

which is the required formula (3).

2. <u>An alternative expression for C(b) and an integral representation of K(s):</u> Define

$$K(s) = \sum_{i=0}^{\infty} C(i)\, s^i$$

and

$$P(s) = \sum_{i=0}^{\infty} S(i)\, s^i$$

Then $bC(b)$ is the coefficient of s^{b-1} in $K'(s)$ and $\sum_{j=0}^{b} S(j)\, C(b-j)$ (with $S(0) = 0$) is the coefficient of s^b in $K(s)\, P(s)$. Therefore Mignosi's recursive formula can be expressed as

$$s\, K'(s) = K(s)\, P(s)$$

or $$K(s) = \exp\left[\int \frac{P(s)}{s}\, ds \right] \qquad (5)$$

$C(b)$ is therefore the coefficient of s^b in $K(s)$ given by (5).

As an application of this integral form consider the case when $a_1 = a_2 = \dots = a_n = 1$. In this case $S(i) = n$ for all $i \geq 1$, so that $P(s) = ns/(1-s)$.

$$K(s) = \exp\left[\int \frac{n}{(1-s)}\, ds\right] = (1-s)^{-n}$$

Hence $C(b) = \binom{n+b-1}{b}$ which is a well known result.

3. <u>A determinant expression for C(b)</u>: The recursive formulas can be expressed as a system of linear equations in $C(1), \dots C(b)$ as

$$AC = S$$

where C is a column vector $(C(1), \dots C(b))$, S is a column vector $(S(1), \dots S(b))$ and the bxb matrix A given by

$$A = \begin{bmatrix} 1 & 0 & 0 & \dots & 0 & 0 \\ -S(1) & 2 & 0 & \dots & 0 & 0 \\ -S(2) & -S(1) & 3 & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & & \vdots & \vdots \\ -S(b-2) & -S(b-3) & -S(b-4) & \dots & (b-1) & 0 \\ -S(b-1) & -S(b-2) & -S(b-3) & \dots & -S(1) & b \end{bmatrix}$$

Note that $A=(a_{ij})$, $a_{ij} = \begin{cases} i & \text{for } i=j \\ -S(i-j) & \text{for } j \; i \\ 0 & \text{for } j \; i \end{cases}$

Since $|A| = b! \neq 0$, $C = A^{-1} S$

4. Computation of C(b) and comparison with upper and lower bounds:

computer programme was written to calculate C(b) from the recursive formula. Padberg [4] and A.G. Beged Dov [1] studied the problem of finding upper and lower bounds on the number N of integral solutions to Knapsack constraint

$$\sum_{i=1}^{n} a_i x_i \leq b \qquad (6)$$

$x_i \geq 0$ and integer.

a_i's and b are integer and > 0. A.G. Beged Dov showed that

$$\frac{b^n}{n! \prod_{j=1}^{n} a_j} \leq N \leq \frac{\left(b + \sum_{1}^{n} a_j\right)^n}{n! \prod_{j=1}^{n} a_j} \qquad (7)$$

Padberg in [4] sharpened the above lower bound to

$$\frac{(b+1)^n}{n! \prod_{j=1}^{n} a_j} \leq N \qquad (8)$$

and gave alternative bounds on N as follows:

$$\binom{n+b^*}{n} \leq N \leq \binom{n+b^{**}}{n} \qquad (9)$$

where b^{*} is the largest integer and b^{**} the smallest integer satisfying

$$b^{*} \le b/a_j \qquad \text{for all } j$$

$$b^{**} \ge [b/a_j] \quad \text{for all } j$$

where $[p]$ denotes the integral part of p.

Formula (9) has the advantage of yielding the exact number of solutions to (6) for the case when all the a_j's are equal. The exact expressions for N developed in this paper serves us to compare these bounds in a few examples. The following nine examples were considered and the exact number of solutions were given by the recursion formula. A table giving the comparison of the bounds with the exact values has also been prepared.

Examples

(a) $5 x_1 + 2 x_2 + 4 x_3 + 3 x_4 + 4 x_5 \le 11$

(b) $x_1 + x_2 + 10 x_3 \le 12$

(c) $3 x_1 + 9 x_2 + 2 x_3 + 11 x_4 + 6 x_5 \le 25$

(d) $11 x_1 + 33 x_2 + 2 x_3 + 7 x_4 + 19 x_5 \le 89$

(e) $5 x_1 + 7 x_2 + 11 x_3 + 13 x_4 + 17 x_5 + 19 x_6 + 23 x_7 + 29 x_8 \le 89$

(f) $2 x_1 + 4 x_2 + 3 x_3 + 4 x_5 + 2 x_6 + x_7 + 2 x_8 \le 90$

(g) $55 x_1 + 57 x_2 + 34 x_3 + 32 x_4 + 3 x_5 \le 61$

(h) $11 x_1 + 13 x_2 + 23 x_3 + 57 x_4 + 6 x_5 + 19 x_6 \le 97$

(i) $3 x_1 + 2 x_2 + 5 x_3 + 7 x_4 + 3 x_5 \le 87$

Table: Comparison of the bounds with the Exact Values

Example	Exact number of solutions	Formula (7) Lower Bound	Formula (7) Upper Bound	Formula (9) Lower Bound	Formula (9) Upper Bound
(a)	53	5	356	21	252
(b)	97	37	230	4	455
(c)	236	28	1287	21	6188
(d)	2460	510	9335	21	1.9×10^6
(e)	4942	100	97450	165	1.08×10^6
(f)	4.9×10^7	2.67×10^7	8.85×10^7	1.56×10^6	1.28×10^{10}
(g)	46	1	675	6	53130
(h)	936	58	8659	7	74613
(i)	115560	69806	185523	6188	1.71×10^6

It is clear from the table that, formula (9) gives very poor results in case where a_j differ very much. Even when they vary in a small range as in example (f) (a_i's vary between 1 and 4) formula (9) need not give closer bounds. In general for these examples formula (7) seems to be better than formula (9). Even formula (7) does not fare well in examples (a), (c), (e), (g) and (h).

Acknowledgment

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