Technical Report

STOCHASTIC POINT PROCESSES IN
A STORAGE MODEL

A. Rangan and A.H.Kalro

विद्याविनियोगाद्विकासः
IIM
AHMEDABAD

INDIAN INSTITUTE OF MANAGEMENT
AHMEDABAD

STOCHASTIC POINT PROCESSES IN A STORAGE MODEL

A. Rangan and A.H.Kalro

T R No. 49

September 1974

Indian Institute of Management
Ahmedabad

No. 41

To

Chairman (Research)
IIMA

Technical Report

Title of the report: Stochastic Point Processes in a Storage Model

Name of the Author: A Rangan and A H Kalro

Under which area do you like to be classified? P&QM Area

ABSTRACT (within 250 words)

A warehouse with a storage capacity of n units is considered. While the units of input arrive according to a general stationary point process $\phi(t)$, the stored items are cleared with the arrival of a bulk order which is assumed to arrive according to another general stationary point process $\Psi(t)$. When the inputs exceed n, there is a cost associated with an emergency clearance. Further there are costs associated with shortage when the bulk order arrives to find the store empty and costs of maintainance of the n unit warehouse. The techniques of stationary point processes are employed to study this selective interaction and to arrive at the total cost function at any time t. This total cost function is used to obtain the optimal warehouse capacity n.

Please indicate restrictions if any that the author wishes to place upon this note: nil

Date: 29 August 1974

Signature of the Author

STOCHASTIC POINT PROCESSES IN A STORAGE MODEL

A. Rangan and A. H. Kalro

Abstract

A warehouse with a storage capacity of n units is considered. While the units of input arrive according to a general stationary point process $\phi(t)$, the stored items are cleared with the arrival of a bulk order which is assumed to arrive according to another general stationary point process $\psi(t)$. When the inputs exceed n, there is a cost associated with an emergency clearance. Further there are costs associated with shortage when the bulk order arrives to find the store empty and costs of maintenance of the n unit warehouse. The techniques of stationary point processes are employed to study this selective interaction and to arrive at the total cost function at any time t. This total cost function is used to obtain the optimal warehouse capacity n.

1. Introduction

The inventory problems arising in different disciplines have received considerable attention in the past few years and an enormous amount of literature has appeared, stimulated by the pioneering papers of Arrow, Harris and Marschak [1], and Dvoretzky, Kiefer and Wolfowitz [2], [3]. A systematic study of the mathematical theory of inventory and production has been made by Arrow, Karlin and Scarf [4]. Useful reviews have been given by Gani [5] and Vienott [6].

A variety of models have been developed to represent inventory problems in different contexts, but the basic structure of any inventory problem remains unchanged. Exogeneous and endogeneous demands made upon a system is satisfied by outputs from inventory, and the inventory in turn is replenished by inputs. In general, if for an item during time period t (discrete time units being considered), the input is X_t and output is Y_t, then the accumulated inventory I_T at time period T is given by

$$I_T = I_o + \sum_{t=1}^{T} (X_t - Y_t) \quad (T = 1,2, \ldots\ldots) \tag{1}$$

where I_o = inventory at time 0 (reference point). I_T may be positive, denoting stocks on hand or negative denoting unfilled orders (in the case of back ordering). If time is considered as a continuous variable, we have

$$I(t) = I(o) + \int_o^t \left[X(\tau) - Y(\tau) \right] d\tau \tag{2}$$

where $X(\tau)$ is the input rate and $Y(\tau)$ is the output rate.

One must however identify the difference between the output rate $Y(\tau)$ and demand rate $\phi(\tau)$. The two are not identical in all cases if we take into account shortages or stockouts. The objective in any given situation is to take that set of decisions (order rules, replenishment quantities, reorder points, etc.) which will minimize total costs or maximize profits (their expectation or some other criterion in stochastic cases) and still provides an acceptable level of service.

Most of the models studied hitherto develop these aspects attaching costs and penalties for different quantities of interest. Further they either assume the input to be random while the output is deterministic such as in grain storage and dam theory models; or the input to be deterministic and the output stochastic as in most production and inventory models.

In this article, we assume both the input and output processes as two general stationary point processes. Using appropriate penalty and maintenance costs, we have arrived at the total cost function. This has been optimized to obtain the optimal warehouse capacity. It is interesting to note that such a model has been employed to study the mechanism of formation of neuronal spike trains on the basis of selective interaction between two processes, excitatory and inhibitory [7].

2. The Storage Model

Consider a warehouse with a storage capacity of n units. Inputs to this warehouse are assumed to arrive according to a general stationary point process $\phi(t)$ and bulk orders are assumed to arrive according to another general stationary point process $\psi(t)$. The arrival of a bulk order completely takes away the accumulated units stored. It is assumed that each bulk order triggers off a sequence of inputs and stops the sequence initiated by the previous bulk order. If during the course of accumulation, more than n units get accumulated, an emergency clearance is called for which results in some additional cost for each such clearance. When a bulk order arrives to find an empty store, a penalty cost is incurred. Further there is a cost associated with the maintenance of the n unit warehouse. Since we wish to use the total cost function to obtain the optimal warehouse capacity n, the various components of the total cost function over a fixed time horizon are first derived.

Consider the time origin at a point where an emergency clearance has taken place. Since there are penalty costs associated with the shortages and emergency clearances, let us obtain their mean number over a fixed time horizon.

Let $h_1(t)\,dt$ be the probability of an emergency clearance in $(t, t + dt)$. Since the emergency clearnace in $(t, t + dt)$ should be the first one or the subsequent one counted from the time origin, $h_1(t)$ satisfies the equation,

$$h_1(t) = g_1^{(n)}(t) \int_t^{\infty} p^i(u)\,du + \int_0^t F(u)\,\mathcal{X}(t-u)\,g_1^{(n)}(t-u) \qquad (3)$$

where $\mathcal{X}(t) = \int_t^{\infty} \psi(u)\,du.$

$g_1^{(n)}(t)$ = product density of degree one of a renewal process whose interval distribution is the n-fold convolute of $\phi(t)$

$F(u)du$ = product density of degree one of a bulk order arriving in $(u, u+du)$ starting with an emergency clearance at $u = 0$.

$p^i(t)$ = probability frequency function between an emergency clearance and the next bulk order

and $\phi_n(t)$ = n-fold convolute of $\phi(t)$.

$F(u)$ satisfies the equation

$$F(u) = p^i(u) + \int_0^u p^i(t)\,f_1(u-t)\,dt \qquad (4)$$

where $f_1(t)$ is the product density of degree one of a bulk order given that a bulk order has arrived at $t=0$ whose transform solution is given by $\frac{\psi^*(S)^*}{1-\psi^*(S)}$

To obtain $p^i(t)$ we observe that the origin is not an arbitrary point for the bulk orders in view of the fact that the event at t=0 is an emergency clearance thereby implying that the previous bulk order has arrived prior to n prior inputs. If the earlier bulk order arrives at x units before the origin (this event occurs with probability $-\frac{1}{\Psi'(0)}$ in view of the stationarity where $\Psi'(0)$ is the derivative of the L.T. of $\Psi(x)$ at s = 0), an emergency clearance takes place at t = 0 with probability $G_1^n(x)$ where $G_1^n(x)$ stands for the modified product density of degree one of a renewal process whose interval distribution is the n-fold convolute of $\phi(t)$. Thus the joint probability that the input at t = 0 yields an emergency clearance and the next bulk order arrives between (t, t + dt), t being measured from the origin is given by

$$-\frac{1}{\Psi'(0)} \int_0^\infty \Psi(t+x)\, G_1^n(x)\, dx.$$

$p^i(t)$ is obtained by dividing this by the probability that an emergency clearance takes place at t = 0 so that

$$p^i(t) = \frac{\int_0^\infty \Psi(t+x)\, G_1^n(x)\, dx}{\int_0^\infty \chi(x)\, G_1^n(x)\, dx} \tag{5}$$

$G_1^n(t)$ is obtained from the equation

$$G_1^n(t) = \int_0^t V_F(u)\, \phi_{n-1}(t-u)\, du + \int_0^t G_1^n(u)\, \phi_n(t-u)\, du. \tag{6}$$

where $V_F(u)$ is the forward recurrence time of the inputs.

Now if $N_1(t)$ represents the number of emergency clearances in $(0,t)$ then

$$E_n\left\{N_1(t)\right\} = \int_0^t h_1(\tau)\, d\tau .$$

Let $k_1(t)\,dt$ be the probability of a shortage in $(t, t+dt)$. Keeping in mind that a shortage occurs because a bulk order arrives to find the store empty, which can happen in one of the following three mutually exclusive ways:

i. the bulk order arrives after an emergency clearance, not intercepted by any input;

ii. it arrives after a bulk order, not intercepted by any input; and

iii. it is the first bulk order after $t = 0$, not intercepted by any input.

Thus

$$k_1(t) = p^i(t)\,\eta(t) + \int_0^t h_1(u)\, p^i(t-u)\,\eta(t-u)\, du \qquad (7)$$

$$+ \int_0^t F(u)\,\psi(t-u)\,\eta(t-u)\, du,$$

where $\eta(t) = \int_t^\infty \phi(u)\, du.$

If $N_2(t)$ represents the number of shortages in $(0, t)$, then the mean number or shortages in $(0, t)$ is given by

$$E\left\{N_2(t)\right\} = \int_0^t k_1(\tau)\, d\tau .$$

Let C_1 = Cost of a single emergency clearance

C_2 = Cost of a shortage

C_3 = Cost per unit of operation for a warehouse of size n/unit time

and F (n, t) = The total cost function over the fixed time horizon.

Then

$$F(n; t) = C_1 E_n \left\{ N_1(t) \right\} + C_2 E_n \left\{ N_2(t) \right\} + C_3 nt \qquad (8)$$

Since the optimum warehouse capacity n* is sought for, n* is obtained as the solution of

$$\left.\begin{array}{l} F(n^*+1; t) - F(n^*; t) \geqslant 0 \\ F(n^*-1; t) - F(n^*; t) \geqslant 0 \end{array}\right) \qquad (9)$$

3. A Special Case

The total cost function when both the input and output form two interacting stationary point processes, have been obtained. The equations reduce considerably in length in the special case when both the input and output processes are Poisson with parameters λ and μ respectively. The Laplace transform of $h_1(t)$ reduces in this special to

$$h_1^*(s) = \frac{\lambda^n}{(\lambda+s+\mu)^n - \lambda^n} + \frac{\mu}{s(s+\mu)} \cdot \frac{\lambda^n}{(\lambda+s)^n - \mu^n} \qquad (10)$$

and the Laplace transform of k (t) reduces to

$$k^*(S) = \frac{\mu}{\lambda+S} + \frac{\mu}{\lambda+S} h_1^*(S) + \frac{\mu}{s(\lambda+\mu)} + \frac{\mu}{\lambda+\mu} \frac{1}{(\lambda+\mu+s)} \qquad (11)$$

We wish to observe that resort to numerical methods for the foregoing model with the analytical solution obtained may yield interesting results.

4. An Example

Before concluding this article, we would like to cite an industrial situation wherein the above model may be used with suitable modifications. Consider a machine with N components, each one of which is subject to failure which occurs according to any general point process. The machine can function with (N-n+1) components and there is a total failure of the machine with the failure of the nth component. The servicing mechanism which can service the defective components of the machine is made available to the machine at various instants of time depending on various factors like the number of machines it has to service, the number of machines in operation etc. and thus is taken to arrive according to another general point process. We can allow upto (n-1) components to fail and still have the machine operational. One must determine the optimal number of standby components in a machine such that total relevant costs are minimized.

Acknowledgment

One of the authors would like to gratefully acknowledge the stimulating discussions on "Interacting Renewal Point Processes" in a different context with Professor S.K. Srinivasan, Indian Institute of Technology, Madras.

References

1. Arrow, K., Harris, T. and Marschak, J., "Optimal Inventory Policy," Econometrica, Vol 19, No.3 (July 1951), 250-272

2. Dvoretzky, A., Kiefer, J. and Wolfowitz, J., "The Inventory Problem: I Case of Known Distributions of Demand," Econometrica, Vol. 20, No.2 (April 1952), 187-222.

3. ________________."The Inventory Problem: II Case of Unknown Distributions of Demand," Econometrica, Vol 20, No.3 (July 1952), 451-466.

4. Arrow, K.J., Karlin, S. and Scarf, H., "Studies in the Mathematical Theory of Inventory and Production," Stanford University Press(1958)

5. Gani, J., "Problems in Probability Theory of Storage Systems," J. Rox. Stat. Soc., B 19, (1957), 181-206.

6. Veinott, A.F., "The Status of Mathematical Inventory," Management cience, Vol 12, No.11, (July 1966), 745-777.

7. Srinivasan, S.K., Rajamannar, G. , and Rangan, A., "Stochastic Models for Neuronal Firing," Kybernetik, 8, (1971), 188-193.