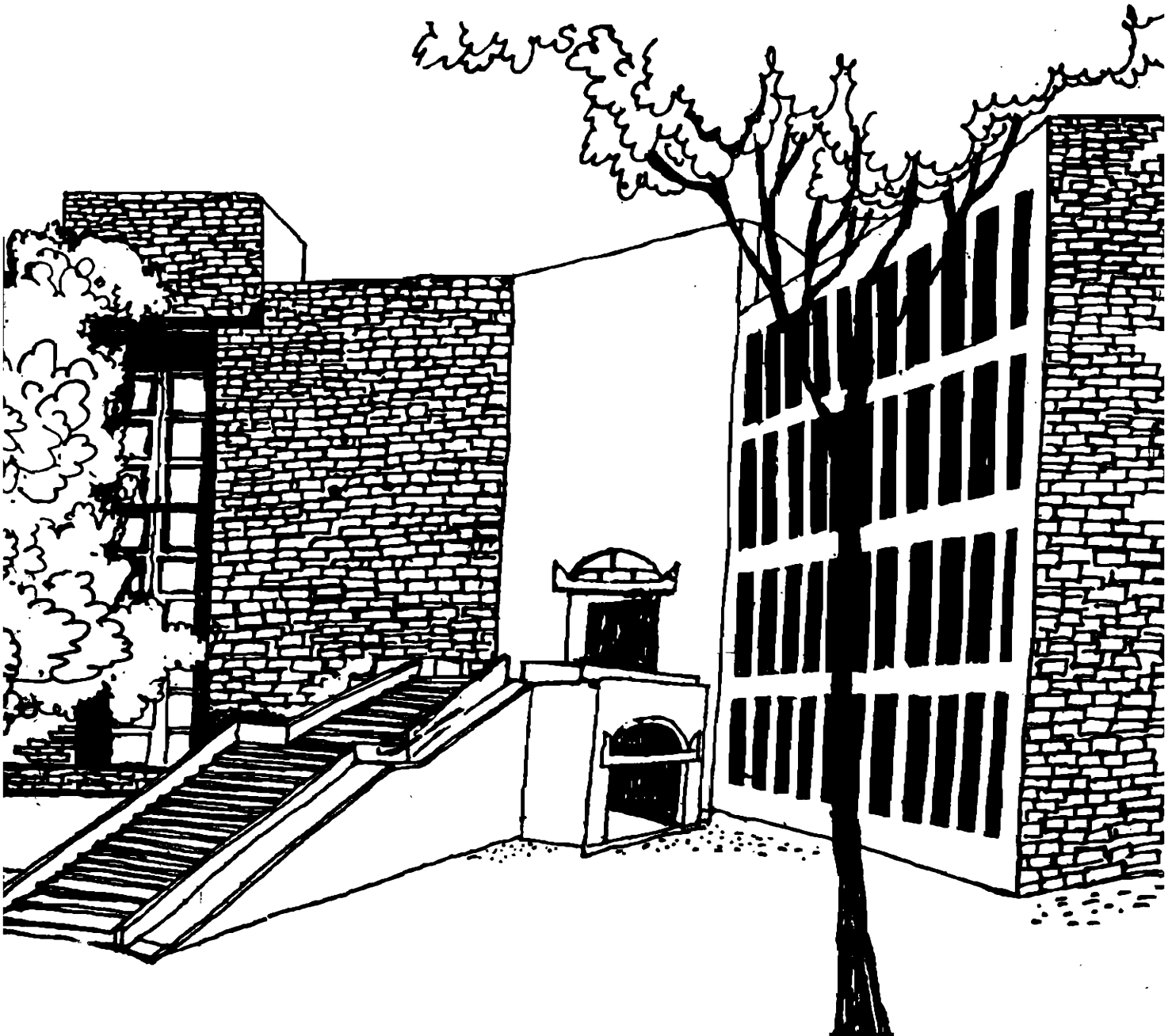




Working Paper



Portfolio Allocation with Heavy-tailed returns

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In Portfolio Allocation with Heavy-tailed returns

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Abstract

In this paper we propose two new methods of portfolio allocation which are applicable for all return distributions. The properties of these new methods are compared with that of Markowitz's mean-variance method using extensive simulation. It is found that the new methods perform appreciably in terms of growth of wealth as well as protecting against the downside risk, in situations where the return distributions of one or more of the stocks is heavy-tailed. These methods can be effective substitutes for the mean-variance method which is not applicable for return distributions with heavy-tails having infinite expectation or variance.



Keyword and Phrases: Portfolio allocation, Mean-variance method, Heavy tailed return distribution

1. Introduction

The aim of modern portfolio theory (Markowitz [1952]) is to optimally allocate assets which would yield maximum return subject to a certain level of risk specified by the investor. Markowitz defined risk as the standard deviation of the distribution of returns from the portfolio. In a more general setting one may consider the investor to maximize his utility, which is a function of both return and risk. Subsequent to Markowitz's seminal work, the problem of determining the optimal allocation of resource to various assets in a portfolio has been extensively studied.

Markowitz's analysis of the portfolio allocation problem is restricted to those assets whose distributions of returns have finite expected values and finite standard deviations. However, it has been observed that distributions of returns of some stocks do not have finite expectation or finite standard deviation. In such cases the Markowitz's mean-variance approach breaks down completely.

In the mean-variance framework, the optimal portfolio allocation among a set of stocks can be done by formulating quadratic programming models which maximize the utility of the investor. The optimal portfolio is obtained by maximizing a utility function, which is a function of the expected return and the variance of the portfolio. Let the random variable r_i denote the return of stock S_i in one time period. The following function is commonly used in the literature on optimal allocation of portfolios:

$$F(x_1, x_2, \dots, x_n) = \sum_{i=1}^n x_i R_i - \alpha \left(\sum_{i=1}^n \sigma_i^2 x_i^2 + 2 \sum_{i=1}^n \sum_{j=i+1}^n \sigma_{ij} x_i x_j \right) \quad \dots(1)$$

where $R_i = E(r_i)$ is the expected return from stock S_i and σ_i^2 is the variance of the returns of stock S_i , σ_{ij} is the covariance of the returns of stocks S_i and S_j , $x_i (\geq 0)$ is the fraction of total investment invested in stock S_i , $\sum_{i=1}^n x_i = 1$ and α is the risk aversion factor of the investor, $0 \leq \alpha < \infty$. Note that large α emphasizes risk minimization and a lower α value emphasizes return maximization. The optimal portfolio is obtained by solving the following quadratic programming problem:

$$\text{Maximize } F(x_1, x_2, \dots, x_n) \text{ subject to } \sum_{i=1}^n x_i = 1, \quad 0 \leq x_i \leq 1 \quad \dots(2)$$

If the utility function is taken to be the Constant Elasticity of Substitution (CES)

utility function $u(r_p) = \frac{e^{-2\lambda r_p} - 1}{-2\lambda}$ where r_p is the return of the portfolio whose

distribution is assumed to be $\text{Normal}(R_p, \sigma_p^2)$ where $R_p = E(r_p) = \sum_{i=1}^n x_i R_i$

and $\sigma_p^2 = V(r_p) = \sum_{i=1}^n \sigma_i^2 x_i^2 + 2 \sum_{i=1}^n \sum_{j=i+1}^n \sigma_{ij} x_i x_j$ then it can be shown that

maximizing the function F is equivalent to maximizing the expected utility $E(u(r_p))$ (Uysal et. al. [2001]). A comprehensive account on portfolio selection is the book by Wang and Xia [2002].

In this paper we propose two new methods for portfolio allocation which works in all situations and study their performance through extensive simulation. In

section 2 we introduce the N-method and in section 5 the D-method. We study the behaviour of these methods of portfolio allocation in several situations including some where the mean-variance method is applicable. We find that the N-method performs much better than the mean-variance method in terms of Value-at-Risk¹ (VaR) while giving reasonable performance in terms of wealth maximization. The D-method is seen to give better returns at the same VaR as that of the mean-variance method. The results of this paper will be helpful for investment bankers and financial analysts in determining the optimal portfolio allocation in case of stocks with non-normally distributed returns.

The paper is organised as follows : in section 2 we introduce the N-method, in section 3 we discuss the performance of the N-method using simulation, in section 4 we discuss an application to Indian stock markets, in section 5 we introduce the D-method and discuss its performance and in section 6 we make some concluding remarks.

2.0 New Methodology

In situations where the probability distribution of the returns of one or more of the stocks does not have a finite expectation, it is not possible to obtain the optimal allocation by maximizing (2) since F is not even defined. This is an unhappy situation since in real life situations portfolio managers usually deal with hundreds or even thousands of stocks and need to decide on an optimal portfolio based on these stocks. In such situations it is unrealistic to expect

¹ The usual definition of Value-at-Risk (VaR) of a given portfolio of instruments is the maximum expected loss for a given time horizon and for a given confidence level, attributable to changes in the market price of financial instruments.

that the distribution of returns of all the stocks under consideration will have finite expectation and variance. It is thus worthwhile to search for a criterion which is defined in all situations and which retains the basic features of risk-return trade-off.

We now introduce a new criterion which is analogous to (2) but is defined for all probability distributions of the returns. Suppose $R_1^M, R_2^M, \dots, R_n^M$ are the median returns of the stocks S_1, S_2, \dots, S_n respectively and let x_i be the fraction of total investment in stock S_i . Let

$$G(x_1, x_2, \dots, x_n) = \sum_{i=1}^n x_i R_i^M - \alpha \text{Median} \left(\sum_{i=1}^n x_i r_i - \text{Median} \left(\sum_{i=1}^n x_i r_i \right) \right) \quad \dots(3)$$

Thus in constructing G , we have replaced the expected return of each stock with its median return and the standard deviation of returns with Median Absolute Deviation (MAD) of returns. We determine the optimal portfolio allocation by maximizing $G(x_1, x_2, \dots, x_n)$ subject to $\sum_{i=1}^n x_i = 1, 0 \leq x_i \leq 1$. In this context it may be worthwhile to note that MAD can be used as an alternative for measuring risk particularly when the distribution of return does not have finite variance. We refer to this method of portfolio allocation as the N-method.

3.0 Performance of the N-Method

We study the performance of N-method of portfolio allocation through extensive simulation. We also compare the performance of the N-method with that of the mean-variance method. For the purpose of this comparative study we consider the problem of optimal allocation when we have two and three stocks.

3.1 Design of the simulation study

In the simulation study we consider three possible symmetric unimodal probability distributions of returns. They are: Normal(N), Double Exponential(DE) and Cauchy(C) distributions. While Normal and Double

Exponential distributions have finite variance the Cauchy distribution does not even have finite expectation.

We now discuss briefly the methodology followed by us to study the performance of the portfolios obtained by following the mean-variance method and the N-method. α is taken to be 1 for both methods. We discuss the specific case of allocation between two portfolios one of which has return distribution which is $N(0,1)$ and the same for the other is $C(0,1)$. The methodology followed in other cases is similar. The R language is used to generate 1000 random variates from $N(0,1)$ and $C(0,1)$ which are taken as the returns (r_{1i}, r_{2i}) on day i from two stocks S_1 and S_2 . For the mean-variance method estimates of mean return and variance of the return are obtained from the sample data for each of the stocks. The returns of the two stocks are assumed to be independent hence the covariance term is taken as 0. The Excel-Solver is then used to obtain the optimal portfolio. For the N-method obtaining the portfolio allocation is a computationally intensive exercise. We develop an Excel-VBA code which uses the Excel-Solver to obtain the portfolio allocation.

The performance of the optimal allocations obtained from the mean-variance approach and the new methodology are then compared using the criteria (a) Growth of Wealth and (b) Value-at-Risk. The details of these measures of comparison are given in section 3.2. The whole exercise is repeated 50 times.

3.2 Measures of comparison

The two criteria used for comparison of the mean-variance method with the N-method are chosen based on practical applications of portfolio allocation. In most cases, a portfolio allocation is done for managing the wealth of an individual or an organization. In such cases the growth of wealth over time is a natural criterion to measure the effectiveness of the portfolio allocation. However the growth of wealth by itself does not free the investor from the nagging thought of how much can he lose at most because of a certain

portfolio allocation decision. VaR is a widely used measure of that amount of loss.

The two criteria of comparison of the two methodologies used in this study are:

- (a) Growth of wealth: Suppose there are n stocks – for our simulation study $n= 2$ or 3 . The portfolio allocations obtained using the mean-variance method and the N-method are both subjected to market movements for 100 consecutive trading days. If x_i is invested in stock S_i and the returns of S_i for the 100 trading days are $r_{i,1}, r_{i,2}, \dots, r_{i,100}$ then the value of the investment in stock S_i at the end of 100 trading days is $x_i(1+r_{i,1})(1+r_{i,2})\dots(1+r_{i,100})$. Hence the value of the portfolio at the end of 100 trading days is $\sum_{i=1}^n x_i(1+r_{i,1})(1+r_{i,2})\dots(1+r_{i,100})$. The returns for the 100 trading days are obtained through simulation from the same probability distributions based on which the allocations were obtained. The total wealth of the portfolio at the end of 100 trading days for the two methods is then compared. The method with higher wealth at the end of 100 trading days is considered as better.
- (b) Value-at-Risk (VaR): The values of the two portfolios obtained by the two methods are tracked for 100 trading days. The 5 percentiles of the distributions of the values of the two portfolios is defined as the VaRs of the two portfolios. The portfolio whose VaR is greater is declared a winner. For example if VaR for the new methodology is -1.28 and that for mean-variance method is -1.50 then the new methodology is considered as better.

3.3 Results of the simulation study

In Table 1 below we give the relative performance of the two methodologies for optimal allocation between two stocks, and in Table 2 we give the same for three stocks having different return distributions. We assume that the return distributions of the stocks are independent. From Table 1 we see that in the

case when both the return distributions are Cauchy the N-method does far better than the mean-variance method both in terms growth of wealth and VaR. This result is expected since in this case the mean-variance method is not applicable. In the case when both the return distributions are normal we find that mean-variance method does better than the N-method in terms of growth of wealth but the N-method does better than the mean-variance method in terms of VaR. It is seen that the N-method gives better performance with regard to VaR than the mean-variance method in most cases. From Table 2 this phenomenon becomes even clearer. In all the cases considered in Table 2 we find that the VaR performance of the N-method is superior than the mean-variance method. The growth of wealth using the N-method is superior in most cases where one or more of the return distributions is Cauchy. Also, it is clear from Table 2 that the mean-variance method performs much better in terms of growth of wealth if all the return distributions have finite variance.

Considering all of above we conclude that the N-method may be preferred over the mean-variance method if (a) an investor prefers a lower VaR and / or (b) one or more of the return distributions is suspected to have infinite variance i.e. the return distributions have fat tails. In cases when all the return distributions have finite variance and growth of wealth is the main consideration then the mean-variance method performs better than the N-method.

Table 1: Comparison of the N-method with the mean-variance method for two stocks

| Distribution of return of Stock 1 | Distribution of return of Stock 2 | Superior Performance of new methodology over mean-variance approach | |
|-----------------------------------|-----------------------------------|---|------|
| | | Growth of Wealth | VaR |
| Cauchy | Cauchy | 84% | 66% |
| Normal | Normal | 8% | 88% |
| Double Exp | Double Exp | 8% | 84% |
| Normal | Cauchy | 94% | 14% |
| Double Exp | Cauchy | 42%* | 100% |

* The test $H_0: p=0.5$ against $H_1: p \neq 0.5$ is not significant at 5% level of significance.

Table 2: Comparison of the N-method with the mean-variance method for three stocks

| Distribution of return of Stock 1 | Distribution of return of Stock 2 | Distribution of return of Stock 3 | Superior Performance of new methodology over mean-variance approach | |
|-----------------------------------|-----------------------------------|-----------------------------------|---|------|
| | | | Growth of Wealth | VaR |
| Cauchy | Cauchy | Cauchy | 98% | 84% |
| Normal | Normal | Normal | 3% | 100% |
| Double Exp | Double Exp | Double Exp | 0% | 99% |
| Normal | Cauchy | Cauchy | 80% | 99% |
| Normal | Normal | Cauchy | 4% | 100% |
| Normal | Double Exp | Double Exp | 1% | 100% |
| Normal | Normal | Double Exp | 1% | 100% |
| Cauchy | Cauchy | Double Exp | 42%* | 74% |
| Cauchy | Double Exp | Double Exp | 10% | 100% |
| Cauchy | Normal | Double Exp | 70% | 68% |

4.0 Performance of N-method with real data from Indian stock market

The N-method and the mean-variance method are applied on real Indian stock market data. We take the position of an investor who is willing to invest Rs. 100 in the stock market. We consider the performance of the two methods for two portfolios each having two stocks. Portfolio I comprise of the stocks - Williamson Tea Assam Ltd and Indiabulls and Portfolio II comprises of the stocks - Infosys Technologies and Gujarat Ambuja Cement. The four stocks belonged to different industries / sectors. The correlation matrix of the four stocks is given below:

Table 3: Correlation Matrix of the four stocks

| | Indiabulls | Infosys | Williamson Tea Assam | Gujarat Ambuja Cement |
|-----------------------|------------|---------|----------------------|-----------------------|
| Indiabulls | 1.000 | .341 | .219 | .362 |
| Infosys | .341 | 1.000 | .174 | .363 |
| Williamson Tea Assam | .219 | .174 | 1.000 | .218 |
| Gujarat Ambuja Cement | .362 | .363 | .218 | 1.000 |

*The test $H_0: p=0.5$ against $H_1: p \neq 0.5$ is not significant at 5% level of significance

We see from Table 3 that the returns of the four stocks are not independent. We make the simplifying assumption that the investor is permitted to buy any amount of the stocks including fractional quantities. The optimal allocation of wealth to these stocks are based on the returns of the stocks calculated on the basis of their closing price on National Stock Exchange of India for the period 11 Jan – 4 Jun, 2005 comprising of 101 trading days. Figure 1 gives the daily return of the four stocks during this period. The data for the daily returns of the four stocks during the above mentioned period are subjected to the Kolmogorov-Smirnov (K-S) test individually. The K-S test, at 5% level of significance, did not reject the normality assumption for all of the four return distributions.

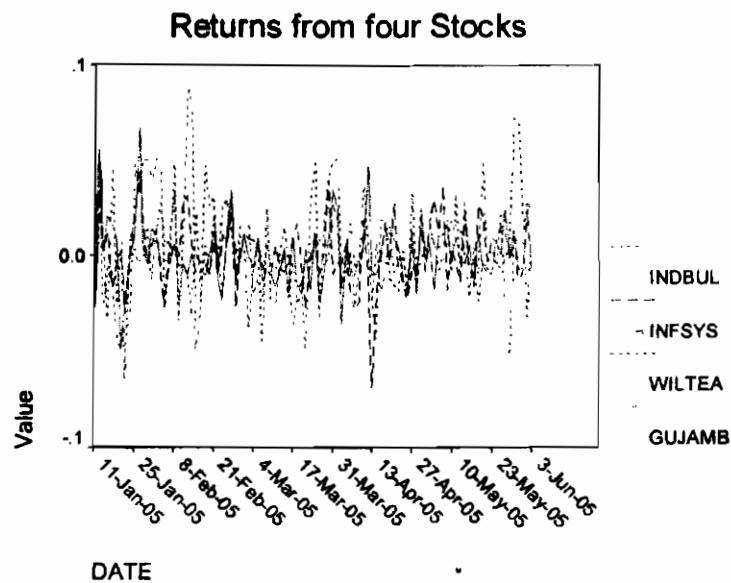


Figure 1 : Daily Returns of the Four Stocks

After the allocation of Rs. 100 is done between the two stocks in a portfolio, we study the movement of the value of the portfolio depending on market movements for a further 49 trading days i.e. up to 12th August, 2005.

We find that the value of the portfolios constructed using the N-method was more resistant against downward market movements than the ones constructed using the mean-variance method. Table 4 gives a comparative

picture of the growth in wealth of the two portfolios for both the methods. While for Portfolio I both the methods posted significant gains, the gains from the N-method is seen to be larger, in the case of Portfolio II the N-method was able to hold the value while the value of the allocation obtained using the mean-variance method was considerably eroded.

Table 4: Comparison of growth of wealth after 50 trading days of the new methodology with the mean-variance approach for portfolios of some Indian stocks

| Portfolio | Stock 1 | Stock 2 | New method Growth of Wealth | Mean-Variance approach Growth of Wealth |
|-----------|-------------------------|--------------------------|-----------------------------------|---|
| I | Williamson Tea Assam | Indiabulls | 92.3% | 24.2% |
| II | Infosys | Gujarat Ambuja Cement | 0.25% | -53.75% |

5.0 Dependent Returns

In this section we propose another method for optimal allocation, which is a straight forward adaptation of the mean-variance method. Instead of considering mean return, standard deviation of the return and the product moment correlation of two returns we consider the median return, MAD of the returns and the fractile correlation of the returns. Let

$$H(x_1, x_2, \dots, x_n) = \sum_{i=1}^n x_i R_i^M - \alpha \left(\sum_{i=1}^n x_i^2 \text{MAD}(x_i) - 2 \sum_{i=1}^n \sum_{j=i+1}^n r_{ij}^f x_i x_j \sqrt{\text{MAD}(x_i) \text{MAD}(x_j)} \right) \quad \dots(4)$$

where R_i^M as before is the median return of stock S_i and r_{ij}^f is the fractile correlation between the returns of the stocks S_i and S_j . The optimal allocation is that (x_1, \dots, x_n) which maximizes H subject to $\sum_{i=1}^n x_i = 1$, $x_i \geq 0$. We call this method of portfolio allocation as the D-method.

We compare the performance of the allocation obtained by the D-method with that obtained from the mean-variance method using simulation. As before we

take $\alpha = 1$. To generate random variates with specified fractile correlation r^f we use the following approach. It is known that for normally distributed variables, the product moment correlation coefficient r , is related to the fractile correlation coefficient r^f as $r = 2\sin\left(\frac{\pi r^f}{6}\right)$. Two random variates z_1 and z_2 are generated from $N(0,1)$ using standard techniques and the vector $Z = (z_1, z_2)$ is formed. The vector Y is defined as $Y = Z Q$ where Q is the Cholesky decomposition matrix of $R = \begin{pmatrix} 1 & r \\ r & 1 \end{pmatrix}$. We define $u_i = -\Phi(y_i)$, $i = 1, 2$ where Φ is the cdf of $N(0,1)$. Now u_1 and u_2 are two uniform random variates with fractile correlation coefficient r^f . From here we can generate random variates x_1 from distribution F_1 and x_2 from distribution F_2 having fractile correlation coefficient r^f by the usual method i.e. $x_i = F_i^{-1}(u_i)$, $i = 1, 2$.

We use the methodology similar to that discussed in section 3.1 to compare the performance of the D-method with that of the mean-variance method. The results are given in Table 5 below:

Table 5: Comparison of D-method with the mean-variance method for two stocks

| Distribution of returns of stock S_1 | Distribution of returns of stock S_2 | Superior performance of D-method | |
|--|--|----------------------------------|------------------|
| | | Growth of wealth | VaR |
| Double Exp | Double Exp | 73% | 45% [#] |
| Cauchy | Cauchy | 72% | 46% [#] |
| Normal | Normal | 83% | 45% [#] |
| Normal | Cauchy | 52% [#] | 50% [#] |
| Cauchy | Double Exp | 45% [#] | 56% [#] |
| Normal | Double Exp | 98% | 31% |

[#] The test $H_0: p=0.5$ against $H_1: p \neq 0.5$ is not significant at 5% level of significance.

From Table 5 we find that the portfolio allocation obtained by D-method provides better growth of wealth than the mean-variance method with comparable VaR as that of the mean-variance method.

6.0 Concluding Remarks

In section 2 and section 5 of this paper we have proposed two new methodologies, the N-method and D-method, for portfolio allocation which are applicable for all kinds of distributions of returns. Through extensive simulation studies we find that the N-method gives excellent results in terms of protecting against downside risk. It also provides better performance in terms of growth of wealth when the return distribution is Cauchy. In two real life applications the N-method performed far better than the mean-variance method in terms of growth of wealth. The D-method is found to provide better returns with same kind of VaR as the mean-variance method. These two methods can be effective substitutes of the mean-variance method in situations where return distributions of one or more of the stocks are found to be heavy-tailed.

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