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EXISTENCE OF NASH EQUILIBRIUM PROGRAMS OF CAPITAL ACCUMULATION UNDER ALTRUISTIC PREFERENCES : THE CONTINUOUS TIME CASE

By

Somdeb Lahiri

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INDIAN INSTITUTE OF MANAGEMENT
AHMEDABAD-380015
INDIA

ABSTRACT

This paper presents a general model of altruistic growth in continuous time, where agents' welfare are assumed separable in current consumption levels and levels of incremental consumption. We prove the existence of Nash Equilibrium Programs of capital accumulation.

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INTRODUCTION :- Our objective in this paper is to analyze the growth of an economy in continuous time over a period spanning many generations when the basic interest of different generations conflict.

The analysis is concerned with a simple one good model of economic growth. In this model, one generation lives at each instant of time, consuming a portion of the capital stock and investing the remainder in production. The process carries on thus from one instant to the next.

Of course, if each generation cares only about its own consumption, it will consume all that it produces, and there will be no growth. Given, however, a sufficient degree of altruism between generation, the economy might be expected to grow.

Attention is focused here as in Kohlberg [1970], Lane and Mitra [1981], Dasgupta (1974 a,b] and Phelps and Pollack [1968] on the case in which each generation displays a limited degree of altruism towards its immediate successor and no altruism at all towards more distant generations. This is, perhaps the simplest setting in which growth is possible and yet different generations have conflicting interests. Our analysis differs from the earlier mentioned endeavours in that ours is embedded in in continuous time. So whereas, the altruims considered by them is defined on the consumption of succeeding generations, our notion of altruism has to do with the instantaneous increment in consumption at each point in time. Whereas the earlier mentioned analyses rely on a preference structure generated by a utility function which treats present and successor generations symmetrically except for a multiplicative factor. and at any point in time, we have no such convenient analogue at our disposal.

The problem that has to be analyzed is that a generation, in bequeathing capital to its heirs, would like to consume all of the capital; yet it realizes that the heirs, out of their altruism to their own heirs, would not do so. The analysis is carried out by introducing a game theoretic equilibrium notion which we call an "equilibrium consumption schedule". Whereas Kohlberg (1976) uses consumption schedules which are possibly non-linear Dasgupta(1974a,b) and Lane and Mitra(1981) use consumption schedules which are linear with zero intercept. We restrict our strategy space to consumption schedules of the latter variety.

In section 1, we study the production structure of the model and introduce notions which are central to our analysis. In section 2, we specify the intertemporal preference structure under altruism that would hold in continuous time. In section 3, we discuss the concept of Nash equilibrium relevant to our model. In section 4 and 5 we establish the existance of a Nash equilibrium program of capital accumulation when the utility function is assumed to be concave.

1. Production: We consider a one-good economy, with a technology given by a function f, from R_+ to itself. The production possibilities consist of inputs k, and levels of investment $\dot{k}$.

The following assumption on f is maintained throughout:

(F) $f(0) = 0$, and for $k \geqq 0$, f is strictly increasing, concave and twice differentiable.

We consider the initial input level, $\underline{k}$ to be historically given and positive.

A <u>feasible input program</u> is a continuously differentiable function $k : (0, +\infty) \longrightarrow R_+$ denoted $\langle k(.)\rangle$ satisfying

(1) $k(0) = \underline{k}$, $0 \leqq k(t) \leqq f(k(t))$ for $t \geqq 0$, $\dot{k}(t) \leqq \bar{I}$ ∀ $t \geqq 0$

(2) $\dot{k}(.) : (0, +\infty) \longrightarrow R$ is locally Lipschitz

By a theorem of Rademacher, Federer(1969), a locally Lipschitzian function is differentiable at almost every interior point of $(0, +\infty)$.

The consumption program $C : (0, +\infty) \longrightarrow R_+$ denoted $\langle c(.)\rangle$ generated by $\langle k(.)\rangle$ is

(3) $C(t) = f(k(t)) - \dot{k}(t)$ $(\geqq 0)$, for $t \geqq 0$.

The pair $\langle k(.), \dot{k}(.)\rangle$ is called a <u>production program</u> and in addition when (1) is satisfied it is called a <u>feasible</u> program, it being understood that $\langle k(.)\rangle$ is an <u>input-program</u>, $\langle k(.), \dot{k}(.)\rangle$ is a <u>feasible production program</u>, and $\langle C(.)\rangle$ the corresponding <u>consumtption program</u>.

Given a historically defined level of initial input $\underline{k}$, and an <u>investment program</u> $\dot{k}(.) : [0, +\infty) \rightarrow \mathbb{R}$ denoted $\langle \dot{k}(.)\rangle$, there is associated

(i) a unique input program $k(.) : [0, +\infty) \rightarrow \mathbb{R}_+$ defined by

$$k(t) = \underline{k} + \int_0^t \dot{k}(s)\, ds$$

(ii) a unique consumption program $C(.) : [0, +\infty) \rightarrow \mathbb{R}_+$ defined by

$$C(t) = f\left(\underline{k} + \int_0^t \dot{k}(s)\, ds\right) - \dot{k}(t)$$

and conversely.

We shall carry out a significant part of our analysis using <u>investment programs</u>, instead of the associated feasible program.

A feasible program $\langle k(.), \dot{k}(.), C(.)\rangle$ is called <u>interior</u> if $C(t) > 0$ for $t \geq 0$.

The consumption ratio function $z(.)$ associated with a feasible program $\langle k(.), \dot{k}(.), C(.)$ is given, for $t \geq 0$ by

(4) $$z(t) = \frac{c(t)}{f(k(t))} \quad \text{if } f(k(t)) > 0$$

$$= 0 \quad \text{if } f(k(t)) = 0$$

In terms of the associated investment program $\dot{k}(\cdot)$ i.e. the investment program which generates the feasible program $k(\cdot)$, $\dot{k}(\cdot)$, $c(\cdot)$ the consumption ratio function is given, for $t \geq 0$ by

$$z(t) = \frac{f\left(\underline{k} + \int_0^t \dot{k}(s)\,ds\right) - \dot{k}(t)}{f\left(\underline{k} + \int_0^t \dot{k}(s)\,ds\right)} \quad \text{if } f\left(\underline{k} + \int_0^t \dot{k}(s)\,ds\right) > 0$$

$$= 0 \ , \text{ if } f\left(\underline{k} + \int_0^t \dot{k}(s)\,ds\right) = 0$$

The corresponding saving-ratio function is $\langle s(\cdot)\rangle = \langle \quad \rangle$ $\langle 1 - z(\cdot)\rangle$. A feasible program $\langle k(\cdot), \dot{k}(\cdot), c(\cdot)\rangle$ is stationary if $z(t) = \bar{z}$ for $t \geq 0$.

2. Preferences : Individuals are considered to be identical except for their dates of birth. The group of individuals born at the beginning of period 't' is called the tth generation. Each generation lives for precisely one period, and is replaced by an equal number of direct descendants the instant they die.

The preferences of each generation are the same, and are representable by a welfare function U from $\mathbb{R}_+ \times \mathbb{R}$ to $\mathbb{R}_*$ (the extended reals). We consider generation t's welfare, denoted by $u(t)$, to be dependent on its own consumption, and on the increment in consumption received by its immediate descendants. Thus, we can associate with a feasible program $\langle k(\cdot), \dot{k}(\cdot), C(\cdot)\rangle$ a welfare function $u(\cdot)$, given by

(5) $$u(t) = U(C(t), \dot{C}(t)) \quad \text{for} \quad t > 0$$

The following assumption of U is maintained throughout :

(U) $U(C, \dot{C}) = V(C) + b\dot{c}$, where V is a function from $\mathbb{R}_+$ to $\mathbb{R}_*$, and $b > 0$.

We refer to V as the utility function of current consumption. Observe that welfare has been assumed to be transferable in that

$$U(C, \dot{C}) = V(C) + b\dot{c}$$

$$= b\left[\frac{V(C)}{b} + \dot{C}\right]$$

$$= V(0) + b\left[\frac{V(C)}{b}\right] + \dot{C} - V(0)$$

$$= U(0)$$
$$= U\left(C, \frac{V(C)}{b} + \dot{C}\right) - V(0),$$

ie. welfare depends entirely on the co-ordinate representing incremental consumption. We assume that V(.) satisfies.

(V) V(.) is increasing, concave and twice differentiable for $C > 0$.

3. Concept of Nash equilibrium : We assume that each generation can choose its own consumption ratio. Equivalently, each generation can choose a consumption schedule $C(y)$ but it must be linear with zero intercept.

We wish to consider a solution concept which will locate those feasible programs which, in some minimal sense, imply a ''just'' savings principle and so an equitable allocation among generations. To find such a savings principle. Rawls proposed the hypothetical contruct of the ''original position''. All generations, past, present and future, belong to the original position at any time t; however, each generation is assumed ignorant as to when it will exist. A member of the original position is required to assume it is born at some arbitrary time 't', and to ask if $s(t)$ is the best savings ratio from its point of view, on the assurance that all past and future generations live up to their obligations to save according to the schedule $\langle s(\cdot)\rangle$. A prerequisite for $\langle s(\cdot)\rangle$ to be considered ''just'' is that this requirement of individual rationality be met for all generations; furthermore, as t is arbitrary, there would be an internal contradiction if such programs were not intertemporally consistent.

These considerations are captured in the Nash equilibrium solution concept; i.e. for almost every 't', $\bar{z}(t)$ must maximize generation t's utility subject to all other consumption ratios being given by $\bar{z} : [0, +\infty) \to [0,1]$ i.e. by $\langle \bar{z}(.) \rangle$. Formally, a feasible program $\langle \bar{k}(.), \dot{\bar{k}}(.), \bar{c}(.) \rangle$ is a hash equilibrium program if its consumtpion ratio schedule $\bar{z}(.)$ is such that for i.e. $t \geq 0$ and $z \in [0, 1]$

$$(6) \quad U[\bar{z}(t) f(\bar{k}(t)), \dot{\bar{z}}(t) f(\bar{k}(t)) + \bar{z}(t) f'(\bar{k}(t)) \{1 - \bar{z}(t)\} f(\bar{k}(t))]$$

$$\geq U[zf(\bar{k}(t)), \dot{\bar{z}}(t) f(\bar{k}(t)) + zf'(\bar{k}(t)) \{1 - z\} f(\bar{k}(t))]$$

or $$V[\bar{z}(t) f(\bar{k}(t))] + b[\dot{\bar{z}}(t) f(\bar{k}(t)) + \bar{z}(t) f'(\bar{k}(t)) \{1 - \bar{z}(t)\} f(\bar{k}(t))]$$

$$\geq V[zf(\bar{k}(t))] + b[\dot{\bar{z}}(t) f(\bar{k}(t)) + zf'(\bar{k}(t)) \{1 - z\} f(\bar{k}(t))]$$

Let us call this the game $G(\underline{k}, f, U, +\infty)$.

A necessary and sufficient condition for an interior maximum i.e. one which obtains at $z(t) \in (0,1)$ is given by

$$V'(z(t) f(\bar{k}(t))) f(\bar{k}(t)) + b[f'(\bar{k}(t)) \{1 - z(t)\} f(\bar{k}(t)) - z(t) f'(\bar{k}(t)) f(\bar{k}(t))] = 0$$

or $$V'(z(t) f(\bar{k}(t))) f(\bar{k}(t)) + bf'(\bar{k}(t)) f(\bar{k}(t))[1 - 2z(t)] = 0$$

Now we use the crucial fact that all programs introduced so far can be equally well represented in terms of investment programs.

The investment program which generates $\langle \bar{k}(.), \dot{\bar{k}}(.), \bar{C}(.) \rangle$ is $\langle \dot{\bar{k}}(.) \rangle$. A choice of $z(t)$ different from $\bar{z}(t)$ at time t implies a choice $\dot{k}(t)$ of the investment level different from $\dot{\bar{k}}(t)$.

Thus,

$$\left(\frac{V' \; f(\underline{k} + \int_0^t \dot{\bar{k}}(s)\,ds) - \dot{k}(t)}{f(\underline{k} \; \int_0^t \dot{\bar{k}}(s)\,ds)} \cdot f(\underline{k} + \int_0^t \dot{\bar{k}}(s)\,ds)\right)$$

$$f(\underline{k} + \int_0^t \dot{\bar{k}}(s)\,ds)$$

$$+ \, b \, f'(\underline{k} + \int_0^t \dot{\bar{k}}(s)\,ds) \; f(\underline{k} + \int_0^t \dot{\bar{k}}(s)\,ds)$$

$$\left[\frac{1 - 2 f(\underline{k} + \int_0^t \dot{\bar{k}}(s)\,ds) - \dot{k}(t)}{f(\underline{k} + \int_0^t \dot{\bar{k}}(s)\,ds))} \right] = 0$$

if $\dot{k}(t)$ be the investment level at time t which generates a consumption ratio $z(t)$ at time t, which is optimal given the consumption ratio profile $\langle \bar{z}(.) \rangle$.

The above condition simplifies to

$$(7) \quad V'\left(f\left(\underline{k} + \int_0^t \dot{\bar{k}}(s)\,ds\right) - k(t)\right) f\left(\underline{k} + \int_0^t \dot{\bar{k}}(s)\,ds\right)$$

$$+ \; bf'\left(\underline{k} + \int_0^t \dot{\bar{k}}(s)\,ds\right)\left[f\left(\underline{k} + \int_0^t \dot{\bar{k}}(s)\,ds\right) + 2k(t)\right] = 0$$

Thus associated with every investment program $\langle \dot{\bar{k}}(\cdot) \rangle$ is an optimal choice $\dot{\bar{k}}(t)$ at time 't'. $\langle \dot{\bar{k}}(\cdot) \rangle$ is a <u>Nash equilibrium investment program</u> if $\dot{k}(t) = \dot{\bar{k}}(t)$ for a.e. $t \geq 0$.

It should be noted that the choice of $k(t)$ at time 't' depends only on values of $\dot{\bar{k}}(s)$ for $s \leq t$. Thus generation t bases its choice of the optimal investment level on the information it observes upto time 't' and no more.

Given any solution concept, such as the Nash Equilibrium Concept, a natural question to ask is whether the set of Nash Equilibrium is non-empty. This we shall answer in the next two sections. The appropriate discrete-time analogue of this theorem is in Peleg and Yaari(1973). Our approach differs from theirs in three respects. (1) The continuous time version permits a distinction between the stock and flow concept which the discrete time version does not. (2) The continuous time game is played by continuum of agents, where as the discrete time game is played by at most a denumerable infinity of agents. (3) On grounds of methodology the proof leading to existence of Nash equilibrium in the present context differs from the aforementioned endeavour.

4. Finite Time Horizon:-

In this section we consider the situation, where the game lasts over a finite time horizon, i.e., is played over the interval $[0, T]$. Let us call this the game $G(\underline{k}, f, U, T)$.

We prove that for any $\underline{k}$ and any T, there exists a consumption ratio profile $\langle \bar{z}(.) \rangle$ and an associated input program $\langle \bar{k}(.) \rangle$ such that $\langle \bar{k}(.), \dot{\bar{k}}(.), \dot{\bar{z}}(.) f(\bar{k}(.)) \rangle$ is a Nash Equlibrium over the interval $[0, T]$.

Define the following family of functions .

$$B_{Li}([0, T]) = \left\{ g \in C([0,T]) / 0 \leq g(t) \leq \bar{I} \text{ and } |g(t) - g(s)| \leq \bar{I} \; |(t - s)| \;\; \forall t, s \in [0,T] \right\}$$

where $([0,T])$ is the family of continuous, bounded functions on $[0,T]$. Thus the family B_{Li} is bounded by a common bound and is ''equilipschitz'', i.e. all functions of the family share the same Lipschitz constant.

<u>Lemma 4.1</u> : $B_{Li}([0, T])$ is a Convex, Compact Subset of $C([0, T])$.

<u>Proof</u> : We make use of Arzela Ascoli theorem (see Dunford and Schewartz [1957], Ch.4) that states that if M is compact then a set in $C(M)$ is conditionally compact if and only if it is bounded and equicontinuous. Let $M = [0, t]$ and let $C(M)$ be $B_{Li}([0, T])$. Since equilipschitz implies equicontinuity of B_{Li}, Arzela Ascoli theorem can be applied and so B_{Li} is conditionally compact.

Furthermore, by applying the triangle inequality it is clear that B_{Li} is closed since a converging sequence of equilipschitz functions converges to a Lipschitz function with the same constant. Convexity can be shown in the same fashion.

For every investment program $\langle \dot{k}(.) \rangle$, consider choices $\dot{k}(t)$ which satisfy equation (7) :

$$(8) \qquad V'\left(f\left(\underline{k} + \int_0^t \dot{k}(s)\,ds\right) - \dot{\tilde{k}}(t)\right) f\left(\underline{k} + \int_0^t \dot{k}(s)\,ds\right) + b\, f'\left(\underline{k} + \int_0^t \dot{k}(s)\,ds\right)\left[f\left(\underline{k} + \int_0^t \dot{k}(s)\,ds\right) + 2\dot{\tilde{k}}(t)\right] = 0$$

<u>Lemma 4.2</u> : Consider a correspondence $\Phi : B_{Li}([0, T]) \twoheadrightarrow B_{Li}([0, T])$ such that $\dot{\tilde{k}}(t) = \Phi(k(.))(t)$ whenever $\dot{\tilde{k}}(t)$ satisfies equation (8). Under the above assumption the correspondence Φ is a non-empty, upper semicontinuous with respect to the supremum metric $\| g - h \| = \sup_t |g(t) - h(t)|$

<u>Proof</u> : Let $(\langle \dot{k}^{(n)}(.) \rangle)$ be a given sequence of functions in $B_{Li}([0, T])$ and let $(\langle \dot{\tilde{k}}^{(n)}(.) \rangle)$ be a sequence of solutions to (8), with $\lim_{n \to \infty} \dot{k}^{(n)}(.) = \dot{k}(.)$ in the supremum metric and $\dot{\tilde{k}}^{(n)} \in \Phi(\dot{k}^{(n)}(.))$. Furthermore suppose, $\lim_{n \to \infty} \dot{\tilde{k}}^{(n)}(.) = \dot{\tilde{k}}(.)$ in the supremum metric.

Thus,

$$V' \left(f\left(\underline{k} + \int_0^t \dot{k}^{(n)}(s)\, ds\right) - \dot{k}^{(n)}(t)\, f\left(\underline{k} + \int_0^t \dot{k}^{(m)}(s)\, ds\right)\right.$$

$$+ b\, f'\left(\underline{k} + \int_0^t \dot{k}^{(n)}(s)\, ds\right)\left[f\left(\underline{k} + \int_0^t \dot{k}^{(n)}(s)\, ds\right) + 2\, \dot{\tilde{k}}^{(n)}(t)\right] = 0$$

$$\forall n \in N.$$

Owing to the assumed continuity of all the functions, and the that $\lim_{n \to \infty} \dot{\tilde{k}}^{(n)}(.) = \dot{\tilde{k}}(.)$, $\lim_{n \to \infty} k^{(n)}(.) = k(.)$, we obtain,

$$v' \left(f\left(\underline{k} + \int_0^t \dot{k}(s)\, ds\right) - \dot{\tilde{k}}(t)\, f\left(\underline{k} + \int_0^t \dot{k}(s)\, ds\right)\right.$$

$$+ b\, f'\left(\underline{k} + \int_0^t \dot{k}(s) ds\right)\left[f\left(\underline{k} + \int_0^t \dot{k}(s) ds\right) + 2\, \dot{\tilde{k}}(t)\right] = 0$$

Thus, $\dot{\tilde{k}}(.) \in \Phi(\dot{k}(.))$.

Hence Φ is upper semicontinuous.

It is easy to check from equation (7) that it is convex valued. The nonemptiness of Φ follows as an easy consequence of the Implicit Function Theorem.

Note that Φ is not a reaction function since it is not defined on the strategy space but rather on the space of controls.

__Theorem 1__ : The differential game $G(\underline{k}, f, U, T)$ satisfying assumptions (F) and (V) has a Nash Equilibrium for any initial condition $\underline{k}$.

__Proof__ : We make use of the Kakutani Fixed point theorem which states that if A is a compact convex subset of a locally convex linear topological space then every upper semicontinuous mapping from A into itself which is convex valued has a fixed point. Since $([0, T])$ is a Banach Space, from Lemma 4.1, B_{Li} is a compact convex subset of a locally convex space, from Lemma 4.2 the correspondence Φ is an upper semicontinuous mapping, and thus Φ has a fixed point. This fixed point is a Nash Equilibrium investment program.

The economic interpretation of Theorem 1 is that for every initial condition $\underline{k}$, there exists a strategy profile $\langle z(.)\rangle$ such that : first $z(t)$ is the best response for $\{z(s) : s \; [0, T], \; s \neq t\}$ and second, the induced capital path $\langle k(.)\rangle$ starts at $\underline{k}$.

5. Infinite Time Horizon :

In this section we prove the existence of a Nash solution to the game $G(\underline{k}, f, U, +\infty)$, for every $\underline{k}$. Replication of the finite time horizon proof is not possible. To see this note that we have defined a family of Lipschitz functions $B_{Li}([0, T])$. Then we defined a mapping Φ which, we were able to show, was upper semicontinuous. Using this upper semicontinuity and compactness of $B_{Li}([0, T])$ we were able to make use of the Kakutani fixed point theorem. In the infinite case, $B_{Li}([0, +\infty])$ is not compact. We therefore modify B_{Li} in a way to achieve compactness, and retain upper semicontinuity.

Define the following family of functions

$$\Omega_{Li}([0, +]) = \{ g \in C([0, +\infty]) / g = e^{-rt} h \text{ and } h \in B_{Li}([0, +\infty]) \}$$

where $C([0, +\infty])$ is the family of continuous, bounded functions on $([0, +\infty])$.

Lemma 5.1 : $\Omega_{Li}([0, +\infty])$ is a convex, compact subset of $C([0, +\infty])$.

Proof : We make use of an extension of the Arzela Ascoli theorem which states the following :

Let M be an arbitrary topological space and A a bounded subset of $C(M)$. Then A is conditionally compact if and only if for every

$\varepsilon > 0$ there is a finite collection $E = \{E_1, \ldots, E_n\}$ of sets with union M and points $m_i \in E_i$, $i=1, \ldots, n$, such that for $i=1,\ldots, n$, $\sup_{g \in A} \sup_{m \in E_i} | g(m_i) - g(m) | < \varepsilon$ (see Dunford and Schwartz [1957], Ch.4).

From the definite of B_{Li} ([0, T]) (see section 4) it is evident that due to the fact that B_{Li} is equilipschitz, for every finite T there exists a collection as required. Since the functions in B_{Li} ([0, ∞]) are bounded by $\bar{I}$, for every given $\varepsilon > 0$, let T be such that e^{-rt} $2\bar{I} < \varepsilon$. For this T define the collection E' as $\{E_1, \ldots, E_n, E_{n+1}\}$, where $E_{n+1} = [T, +\infty)$. It is clear that, for $i=1,\ldots, n+1$ and $m_i \in E_i$

$$\sup_{g \in \Omega_i} \sup_{m_i \in E_i} | g(m_i) - g(m) | < \varepsilon$$

and thus Ω_{Li} is conditionally compact. It is cumbersome but straightforward to check that Ω_{Li} is closed and thus it is compact.

Define a correspondence $\Phi : B_{Li}([0,+\infty)) \twoheadrightarrow B_{Li}([0,+\infty))$ as the set of best investment programs for each agent in response to a given investment program, satisfying condition (8). Define a correspondence $\theta : \Omega_{Li} \twoheadrightarrow \Omega_{Li}$ such that for every $g \in \Omega_{Li}$

(9) $$\theta(g) = e^{-rt} \Phi(e^{rt} g)$$

The map θ is well defined since by definition of Q_{Li}, $e^{rt} g \in B_{Li}$. In order to prove its upper semicontinuity we need the following definition and lemma.

__Definition__ : Let $x_n, x_0 \in B_{Li}([0, \quad))$. $x_n \xrightarrow{*} x_0$ if and only if for every finite T, $\sup_{t \leq T} | x_n(t) - x_0(t) | \to 0$ as $n \to \infty$.

__Lemma 5.2__ : $e^{-rt} x_n \to e^{-rt} x_0$ if and only if $x_n \xrightarrow{*} x_0$.

__Proof__ : Clearly if $e^{-rt} x_n \to e^{-rt} x_0$ then for every finite T, $\sup_{t \leq T} | e^{-rt} x_n - e^{-rt} x_0 | \to 0$ as $n \to \infty$ and thus $\sup_{0 \leq t \leq T} | x_n(t) - x_0(t) | \to 0$ as $n \to \infty$. Conversely, since the x_n's are bounded, for every given $\mathcal{E} > 0$ there is a T_1 sufficiently large such $\sup_{t \leq T_1} | e^{-rt} x_n - e^{-rt} x_0 | < \mathcal{E}/2$. Therefore for every $\mathcal{E} > 0$, there is T_1 and sufficiently large N such that for every $n \geq N$, $\sup_{t <} | e^{-rt} x_n - e^{-rt} x_0 | < \mathcal{E}$.

__Lemma 5.3__ : Under the above assumptions, the mapping θ as defined in (9) is upper-semi-continuous with respect to the metric $\| g - h \| = \sup_t | g(t) - h(t) |$.

__Proof__ : For every finite T, the result holds as shown in the finite case.

Thus by an application of Lemma 5.2, we obtain that θ is upper semicontinuous.

__Theorem 2:__ The differential game $G(\underline{k}, U, f, +\infty)$ satisfying assumptions (F) and (V) has a Nash equilibrium solution for any initial condition $\underline{k}$.

__Proof:__ The proof follows the proof of Theorem 1, where Ω_{Li}, Lemma 5.1, 5.3 replaces B_{Li}, Lemma 4.1 and 4.2 respectively.

In passing let us observe that the above existance results would hold if the intensity of altruism or measure of altruism was assumed to be dependent on time, ie., instead of assuming b to be a constant, we could at the expense of some additional notation assume that $b : [0, +\infty)) \longrightarrow \mathbb{R}_+$ was a C^1 function.

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