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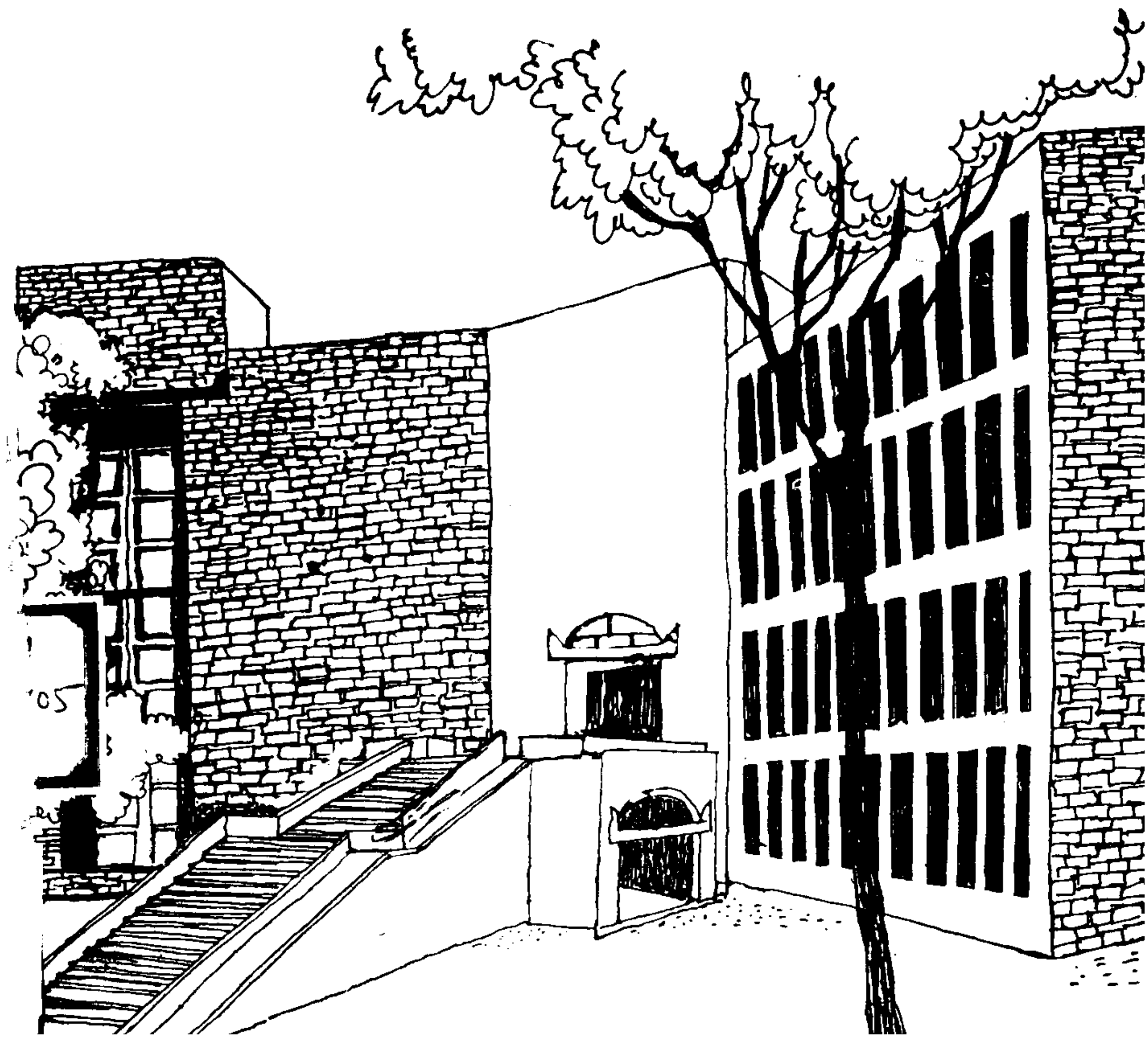

A NEW PROOF OF THE MAXIMUM PRINCIPLE IN OPTIMAL CONTROL THEORY

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ABSTRACT

A new proof of the maximum principle is established in this paper, for the simplest problem in optimal control theory.

The simplest optimal control problem is one of selecting a piecewise continuous control function $u(t)$, $t_0 \leq t \leq t_1$, t_0.

$$\max \int_{t_0}^{t_1} f(t, x(t), u(t))dt \qquad (1)$$

$$\text{Subject to } x'(t) = g(t, x(t), u(t)), \qquad (2)$$

$$t_0, t_1, x(t_0) = x_0 \text{ fixed; } x(t_1) \text{ free} \qquad (3)$$

Here f and g are assumed to be known and continuously differentiable functions of three independent arguments, none of which is a derivative.

It has been proved in the literature on optional control that if the functions $u^*(t)$, $x^*(t)$ maximize (1), subject to (2) and (3), then there is a continuously differentiable function (t) such that u^*, x^*, λ^* simultaneously satisfy the <u>state equation</u>

$$x'(t) = g(t, x(t), u(t)), \; x(t_0) = x_0, \qquad (4)$$

the multiplier equation

$$\lambda^{*\prime}(t) = -[f_x(t, x(t), u(t) + \lambda^*(t) g_x(t, x(t), u(t))], \lambda^*(t_1) = 0 \qquad (5)$$

and the <u>optionality condition</u>

$$f_u(t, x(t), u(t) + \lambda^*(t)\, g_u(t, x(t), u(t)) = 0 \qquad (6)$$

for $t_0 \leq t \leq t_1$. The multiplier equation (5) is also known as the <u>costate</u>, <u>anxilliary</u>, <u>adjoint</u>, or <u>influence</u> equation.

The device for remembering or generating these conditions (similar to solving a non-linear programming problem by forming the Lagrangian, differentiating, etc.) is the Hamiltonian

$$H(t, x(t), u(t), \lambda(t) = f(t,x,u) + \lambda g(t,x,u) \qquad (7)$$

how,

$$\frac{\partial H}{\partial u} = 0 \text{ generates } (6) : \frac{\partial H}{\partial u} = f_u + \quad g_u = 0 \qquad (6')$$

$$-\frac{\partial H}{\partial x} = \lambda^{*'} \text{generates}(5) : \lambda^{*'}(t) = -\frac{\partial H}{\partial x} = -(f_x + \lambda g_x); \quad (5')$$

$$\frac{\partial H}{\partial \lambda} = x' \text{ recovers } (4) : x' = \frac{\partial H}{\partial \lambda} = g \qquad (4')$$

In addition, we have $x(t_0) = x_0$ and $\lambda(t_1) = 0$. At each t, u is a stationary point of the Hamiltonian for the given values of x and λ. One can find u as a function of x and λ from (6) and substitute into (5) and (4) to get a system of two differential equations in x and λ.

For a maximization problem, it is also necessary that u*(t) maximize $H(t, x^*(t), u, \lambda^*(t))$ with respect to 'u'. (See Kamien and Schwartz(1981) page 117). In this paper we provide a new proof of this statement.

Theorem:- Given the optional control problem (1), (2) and (3), a solution $u^*(t), x^*(t)$ satisfies (4),(5),(6) for some continuously differentiable λ^* and in addition $u^*(t)$ maximizes $H(t, x^*(t), u, \lambda^*(t))$ with respect to u.

Proof:-

$$\int_{t_0}^{t_1} f(t,x^*(t),u^*(t))dt = \int_{t_0}^{t_1} f(t,x^*(t),u^*(t))dt + \int_{t_0}^{t_1} \lambda^*(t) \left[g(t,x^*(t),u^*(t)) - x^{*\prime}(t)\right] dt$$

Suggests that the maximization of the left hand side subject to the constraints (2) and (3) is tantamount to the maximization of the right hand side subject to the constraints (2) and (3).

Let $x^*(t)$ be given. Then

$$\int_{t_0}^{t_1} f(t,x^*(t),u(t))dt = \int_{t_0}^{t_1} f(t,x^*(t),u(t))dt + \int_{t_0}^{t_1} \lambda(t)g(t,x^*(t),u^*(t))dt - \int_{t_0}^{t_1} \lambda(t)x^{*\prime}(t)dt$$

$$= \int_{t_0}^{t_1} [f(t,x^*(t),u(t)) + \lambda(t)g(t,x^*(t),u(t))]dt - \int_{t_0}^{t_1} \lambda(t)x^{*\prime}(t)dt$$

whenever the constraint (3) holds. Subject to the constrant (3), the maximization of the left hand integral with a suitable choice of u(t) is equivalent to maximizing the first integral on the right hand side again with a suitable choice of $u(t)$, provided $\lambda(t)$ is given. Assume $\lambda(t_1) = 0$ for all λ.

For every $\lambda(t)$ (a continuously differentiable function) associate the piecewise continuous control $u_\lambda(t)$ which maximizes,

$$\int_{t_0}^{t_1} [f(t,x^*(t),u(t)) + \lambda(t)g(t,x^*(t),u(t))]\, dt$$

Note the controller so chosen depends on λ. Now, choose that λ which satisfies the constraint with equality. Such a λ exists. Infact $\lambda = \bar{\lambda}$ (ie. the multiplier function which satisfies

$$x^{*\prime}(t) = g(t,x^*(t),u_{\bar{\lambda}}(t)), \quad x(t_0) = x_0$$

will do.

Since what has been written above is a simple calculus of variations problem in u, with x^* and $\bar{\lambda}$ now known, a necessary condition is that the familiar Euler equation be satisfied. This implies that

$$f_u(t,x^*(t), u_{\bar{\lambda}}(t)) + \bar{\lambda}(t)g_u(t,x^*(t), u_{\bar{\lambda}}(t)) = 0 \qquad (8)$$

It remains to verify that $\bar{\lambda}(t)$ and $u_{\bar{\lambda}}(t)$ satisfies the familiar costate equation. However, observing that $x^*(t)$, $\bar{\lambda}(t)$ and $u_{\bar{\lambda}}(t)$ are also the solution of (1) subject to (2) and (3), and substituting (8) into the equation of first variation of the optional control problem ie.

$$0 = \int_{t_0}^{t_1} \{(f_x + \lambda g_x + \lambda')y_a + (f_u + \lambda g_u)h\} \, dt$$

implies $\bar{\lambda}$ satisfies the costate equation and is equal to λ^* by uniqueness of solution. Here h is the perturbation in the control, y_a is the first derivative of the associated perturbation in the state variable (See Kamien and Schwartz(1981) page 115).

So $u_{\bar{\lambda}}(t) = u_{\lambda^*}(t) = u^*(t)$ by our choice of notation.

Hence $u^*(t)$ maximizes,

$$\int_{t_0}^{t_1} H(t, x^*(t), u(t), \lambda^*(t))\, dt$$

Suppose for some $\bar{t} \in (t_0, t_1)$, there was a $\bar{u}(\bar{t})$ such that

$$H(\bar{t}, x^*(\bar{t}), \bar{u}(\bar{t}) \;\; *(\bar{t})) > H(\bar{t}, x^*(\bar{t}), u^*(\bar{t}), \lambda^*(\bar{t}))$$

By continuity this will be true on an ϵ neighbourhood of $\bar{t}$

Redefine, $\bar{\bar{u}}$ as follows:

$$\bar{\bar{u}}(t) = u^*(t) \text{ for } t \in [t_0, t_1],\; t \notin [\bar{t}-\varepsilon, \bar{t}+\varepsilon]$$

$$= \bar{u}(t) \text{ for } t \in [\bar{t}-\varepsilon, \bar{t}+\varepsilon]$$

Clearly,

$$\int_{t}^{t_1} H(t,x^*(t),\bar{\bar{u}}(t),\lambda^*(t))dt > \int_{t_0}^{t_1} H(t,x^*(t),u^*(t),\lambda^*(t))dt$$

which is a contradiction.

This proves the theorem.

Appendix:-

Here we show that $f_u + \bar{\lambda} g_u = 0$ implies $\bar{\lambda}' = -f_x - \bar{\lambda} g_x$.

Let $J = \int_{t_o}^{t_1} f(t,x,u)dt$ and let x^* be an optimal solution.

Let $y(a,t) = x^*(t) + ah(t)$, $t \in [t_o, t_1]$, $h(t_o) = 0$ h is continuously differentiable.

Let $u(a,t)$ be such that

$$\frac{d}{dt} y(a,t) = g(t, x(a,t), u(a,t))$$

and $\| u(a,t) - u^*(t) \| = \inf. \{ \| u - u^*(t) \| / \frac{d}{dt} y(a,t) = g(t,x(a,t),u) \}$

$u(a,t)$ exists $\forall\, t \geq 0$, by the compactness of U and continuity of 'g'.

It is continuous in 't' as well.

$$\text{Let } J(a) = \int_{t_o}^{t_1} f(t,y(a,t),u(a,t))dt$$

$$= \int_{t_o}^{t_1} \left\{ f(t,y(a,t),u(a,t)) + \bar{\lambda}(t) \left[-\frac{dy}{dt}(a,t) + g(t,x(a,t),u(a,t)) \right] \right\} dt$$

$$= \int_{t_o}^{t_1} \left\{ f(t,y(a,t),u(a,t)) + \bar{\lambda}(t) g(t,x(a,t),u(a,t)) \right\} dt$$

$$- \bar{\lambda}(t) y(a,t) \Big|_{t_o}^{t_1} + \int_{t_o}^{t_1} \bar{\lambda}'(t) y(a,t) dt$$

$$= \int_{t_o}^{t_1} \left\{ f(t,y(a,t),u(a,t)) + \bar{\lambda}(t) g(t,x(a,t),u(a,t)) + \bar{\lambda}'(t) y(a,t) \right\}$$

$J'(o) = o$ implies

$$0 = \int_{t_o}^{t_1} \Big[\Big\{ f_x(t, x^* \ , u(o,t)) + \bar{\lambda}(t) g_x(t, x^* \ , u(o,t) + \bar{\lambda}'(t) \Big\} h(t)$$

$$+ \Big\{ f_u + \bar{\lambda} g_u \Big\} \frac{\partial}{\partial a} u(o,t) \Big] \ dt$$

$$0 = \int_{t_o}^{t_1} \Big\{ f_x + \bar{\lambda} g_x + \bar{\lambda}' \Big\} h(t) dt \quad \forall \ h \text{ with } h(t_o) = o, \ h$$

continuously differentiable.

$$\therefore \ \bar{\lambda}' = -f_x - \bar{\lambda} g_x$$

REFERENCE:-

1. Kamien, M.I. and N.Schwartz (1981): "Dynamic Optionization: The Calculus of Variations and Optimal Control in Economics and Management," North Holland, New York, Oxford.