T. R. NO.94

Technical Report

WP 1975/94

विद्याविनियोगाद्विकासः
IIM
AHMEDABAD

INDIAN INSTITUTE OF MANAGEMENT
AHMEDABAD

INTERACTIVE EVALUATION AND BOUND PROCEDURE FOR SELECTING MULTI-ATTRIBUTED ALTERNATIVES

by
Rakesh K Sarin

T. R. No. 94
Nov. 1975

Indian Institute of Management
Ahmedabad

To

Chairman (Research)
IIMA

Technical Report

Title of the report ....Interactive Evaluation and Bound Procedure for selecting Multi-attributed alternatives

Name of the Author .....R.K. SARIN.........................

Under which area do you like to be classified? ..Prod'n. & Quant. Methods

ABSTRACT (within 250 words)

.....................................Attached in the first page.

....An Evaluation and bound procedure is developed which............ offers substantial improvement over conventional approach. ....The improvement is measured in terms of the simplicity of the... judgments and the number of judgments that are required from ....from the decision-maker in identifying a preferred decision..... alternative. An extensive experimental study is reported. ....Some experimentally verified rules for implementing the......... procedure on an interactive computer are discussed.

Please indicate restrictions if any that the author wishes to place upon this note ....

URGENT

Date ..13 Oct. '75

R.K. Sarin
Signature of the Author

ABSTRACT

An evaluation and bound procedure is developed which offers substantial improvement over conventional approach. The improvement is measured in terms of the simplicity of the judgments and the number of judgments that are required from the decision-maker in identifying a preferred decision alternative. An extensive experimental study is reported. Some experimentally verified rules for implementing the procedure on an interactive computer are discussed.

ACKNOWLEDGEMENTS

I am very much indebted to Professor James Dyer and Professor Donald Erlenkotter for many helpful suggestions. I am also indebted to Professor Jacob Marschak for several helpful discussion. This paper has been adopted from my doctoral dissertation submitted at University of California, Los Angeles, June, 1975.

In this paper an interactive procedure for the evaluation of multi-attributed alternatives is developed. The procedure is interactive in the sense that it progresses by seeking certain information from the decision-maker. The contribution of the evaluation and bound procedure is in reducing the information burden on the decision-maker, and thus providing practical assistance to him. The reduction in information burden is measured in terms of the simplicity of judgments and the number of judgments that are required from the decision-maker in identifying a preferred alternative. Dyer, Lientz and Sarin [5] discuss an application of the proposed evaluation and bound procedure in the context of Criminal Justice Project Evaluation. In section 1, we define the multi-attributed decision problem. A conventional apporach (see Raiffa [12]) is briefly described in section 2. The evaluation and bound procedure is described in section 3 and an example is given. In section 4, an extensive experimental study is reported. In these experiments the evaluation and bound procedure is applied to randomly generated simulated decision situations. Some experimentally verified rules for implementing the procedure on an interactive computer are also discussed. In section 5, concluding remarks are made and some extensions are suggested.

1. Problem Statement

We define S as the finite set of N decision alternatives, hereafter termed only alternatives. The performance of each alternative in S is measured on n attributes (criteria or objectives). The outcomes or pay offs of the multi-attributed alternatives are represented by the elements in an n-dimensional outcome space X. The set of values or scores that an alternative may possibly obtain on the i^{th} attribute is denoted as X_i. Therefore, the outcome space X is the cartesian product of the X_i, denoted $X_1 \times X_2, \ldots, \times X_n$. x is an element of X denoted by $x = (x_1, x_2, \ldots, x_n)$, where $x_i \in X_i$ for $i = 1, \ldots, n$ is a possible score on the i^{th} attribute. We shall denote $x^1 \succsim x^2$ if the decision-maker either prefers x^1 to x^2 or is indifferent between x^1 and x^2. Our interest is in assessing the utility function over X denoted U(x). The utility function is assessed to facilitate the comparison among multi-attributed alternatives. Suppose, U(x) has been specified and x^1 and x^2 are the outcomes of the alternatives 1 and 2 respectively. Then, by adopting the proposition of utility theory that $x^1 \succsim x^2$ if and only if $U(x^1) \geqq U(x^2)$, the choice between two multi-attributed alternatives is reduced to a single dimensional utility comparison (See Debrue [3] for conditions which guarantee this proposition).

A major portion of the work in the literature has concentrated on the estimation of additive utility functions. Some real life applications of additive utility functions are discussed in [1], [2], [4], [6], and [14]. A utility function is said to be additive when there exist functions U_i ($i = 1, \ldots, n$) such that the following holds:

$$U(x) = U_1(x_1) + U_2(x_2) + \ldots + U_n(x_n) \tag{1.1}$$

for all x in X, where U_i is called a numerical utility function for the i^{th} attribute. For the utility function to be additive certain assumptions about the decision-maker's preferences over X must be satisfied.

In uncertain choice situations, the criterion "expected utility" has been derived as a guide to action under Von Neumann and Morgenstern's [15] conditions or an equivalent set of axioms. In the context of expected utility theory, Fishburn ([7], [9]) has derived necessary and sufficient conditions for a utility function to be additive. Simply stated, Fishburn's condition (commonly known as the "marginality assumption") is that the desirability of any lottery over X should only depend on the marginal probability distributions over the X_i and not on the joint probability distribution. Several tests are discussed in ([7] , [8]) to validate the marginality assumption.

In this paper, we consider the identification of a preferred alternative in the context of the expected additive utility model. However, the procedure developed here is also applicable to multiplicative model as in Keeney [10] (see Sarin [13]). Specifically, the problem can be stated as follows:

Choose $k \in S$ such that $E(U(\underline{x}^k)) \geq E(U(\underline{x}^\ell))$ for all $\ell \in S$

where,

$$(1.2) \qquad E(U(\underline{x}^k)) = \sum_{j=1}^{m} \sum_{i=1}^{n} p_j^k \cdot U_i(x_{ij}^k)$$

for all $k \in S$ and $\underline{x}^k = (\underline{x}_1^k, \underline{x}_2^k, \ldots, \underline{x}_m^k)$ is a risky alternative which has outcome $\underline{x}_j^k$ (an n-vector) when the random event E_j occurs with a probability p_j^k. x_{ij}^k is the score of the alternative k on the i^{th} attribute when the j^{th} event occurs, the i^{th} component of the vector $\underline{x}_j^k$. $E(U(\underline{x}^k))$ is the expected utility of the alternative k, and U_i is the single dimensional utility function for the i^{th} attribute.

2. A Conventional Approach

A preferred alternative can be identified if the probability distribution over the outcomes $\underline{x}_j^k$ and the additive utility function on the attributes can be specified. We assume that the probability distributions for the outcomes of each alternative are known. Raiffa [12] has described an approach for specifying an additive utility function. Below, we describe this approach.

In (1.1) the additive utility function was represented as

$U(x) = \sum_{i=1}^{n} U_i(x_i)$ for all

An alternative representation of the additive utility function is

(2.1) $$U(x) = \sum_{i=1}^{n} w_i \cdot f_i(x_i) \text{ for all } x \in X$$

where,

(2.2) $$U(x_*) = 0, \quad U(x^*) = 1,$$

(2.3) $$f_i(x_{i_*}) = 0, \quad f_i(x_i^*) = 1,$$

(2.4) $$\sum_{i=1}^{n} w_i = 1, \quad w_i \geqq 0,$$

and where $f_i\ (x_i)$ is the conditional utility function for attribute i scaled according to (2.3), and w_i is the scaling factor (importance weight) for attribute i. (2.2) and (2.3) use the following notations:

$$x_i^* \succ x_i^k, \quad k \in S$$

$$x_i^k \succ x_{i_*}, \quad k \in S$$

$$x^* \triangleq (x_1^*, x_2^*, \ldots, x_n^*)$$

$$x_* \triangleq (x_{1_*}, x_{2_*}, \ldots, x_{n_*}).$$

To obtain w_i the decision-maker is asked to make certain indifference judgments between a sure reward and a lottery; e.g. determine π' such that a decision-maker is indifferent between a sure reward $(x_{1_*}, \ldots, x_{i-1_*}, x_i^*, x_{i+1_*}, \ldots, x_{n_*})$ and a lottery in which x^* has π' probability and x_* has $(1-\pi')$ probability of occurrence, denoted $(x^*, x_*; \pi')$. Then $w_i = \pi'$.

Similarly, to obtain the f_i's the following indifference judgment between a sure reward and a lottery is sought:

Determine π'' such that the decision-maker is indifferent between a sure reward $(x_{1_*}, \ldots, x_{i-1_*}, x_i^k, x_{i+1_*}, \ldots, x_{n_*})$

and a lottery $((x_{\bar{i}_*}, x_i^*), x_*; \pi'')$

where $x_{\bar{i}_*} = (x_{1_*}, \ldots, x_{i-1_*}, x_{i+1_*}, \ldots, x_{n_*})$.

Then, $f_i(x_i^k) = \pi''$. It should be noted that the values of all attributes other than the i^{th} attribute can be fixed at any level, and not necessarily at the worst level. By seeking indifference judgments for different x_i's the complete function f_i can be specified.

The utility function constructed above can now be used for probabilistic combinations to compute the expected utility for each alternative.

3. Evaluation and Bound Procedure

In this section we shall develop an evaluation and bound procedure for selecting a multi-attributed alternative. First, the evaluation and bound procedure is described and an example is given. To simplify the exposition only a single event is considered. Next, some generalization of the evaluation and bound procedure are discussed. Finally, it is shown that the proposed procedure is equally applicable for multiple event situations.

3.1 Evaluation and Bound Procedure

The motivation for developing the evaluation and bound procedure was provided by the observation that in the conventional approach, such as described in section 2, the information burden on the decision-maker is considerable. The decision-maker has to provide a large number of indifference judgments. Such judgments place considerable cognitive burden on the decision-maker. The number of such judgments is approximately on the order of $n+n(N-2)$, where n judgments are needed to estimate w_i's, and $n(N-2)$ judgments are needed to estimate $f_i\ (x_i^k)$ for $k = 1$ to N, $i = 1$ to n.

The idea of the evaluation and bound procedure is to simplify the type of judgments as well as to reduce the number of judgments required from the decision-maker. Specifically, the procedure requires only range for the probabilities that determine the w_i and $f_i\ (x_i)$. For example, to determine w_i the decision-maker is asked whether he prefers a sure reward $(x_{1_*}, \ldots, x_{i-1_*}, x_i^*, x_{i+1_*}, \ldots, x_{n_*})$ or a lottery $(x^*, x_*\ ; \pi')$. Suppose the decision-maker prefers the sure reward. Now, a similar question is asked in which the chance of getting x^* in the lottery is π'', $\pi'' > \pi'$.

Suppose now the decision-maker prefers the lottery. Then, $\pi' \leqq w_i \leqq \pi''$, and this information may be sufficient for the evaluation and bound procedure. The indifference judgments needed in the conventional approach may require a series of successively more difficult pairwise judgments like these.

In the evaluation and bound procedure, lower and upper bounds on the utilities of alternatives are established. These bounds on the utilities of alternatives are obtained from the weighted sums of lower and upper bounds on the utilities of scores on the various criteria.

Let: $\underline{B}^k$ = Lower bound on the utility of the k^{th} alternative.

$\overline{B}^k$ = Upper bound on the utility of the k^{th} alternative.

$\underline{f}_i(x_i^k)$ = Lower bound on the utility of the k^{th} alternative's score on the i^{th} criterion.

$\overline{f}_i(x_i^k)$ = Upper bound on the utility of the k^{th} alternative's score on the i^{th} criterion.

Then,

$$(3.1)\qquad \underline{B}^k = \sum_{i=1}^{n} w_i \cdot \underline{f}_i\,(x_i^k),$$

$$(3.2)\qquad \overline{B}^k = \sum_{i=1}^{n} w_i \cdot \overline{f}_i\,(x_i^k),$$

where, $\underline{f}_i\,(x_i^k) \leq f_i\,(x_i^k) \leq \overline{f}_i\,(x_i^k)$. $\underline{B}^k$ and $\overline{B}^k$ are computed for all alternatives $k = 1$ to N.

Given the bounds the procedure tests whether a certain alternative can be ruled out. If

$$(3.3)\qquad \overline{B}^k \leq \underline{B}^{\ell}, \quad \text{then} \quad \ell \succ k.$$

Simply stated, the test determines whether the lower bound of an alternative ℓ is greater than the upper bound of any alternative k. If so, the alternative ℓ is preferred to the alternative k, and the alternative k can be dropped from consideration.

In order to obtain $\overline{B}^k$ and $\underline{B}^k$, we need to estimate w_i, $\underline{f}_i\,(x_i^k)$, and $\overline{f}_i\,(x_i^k)$. w_i is estimated as illustrated in section 2. Latern, in section 3.2 we shall demonstrate that exact estimates of w_i's are not required. To estimate $\underline{f}_i\,(x_i^k)$ and $\overline{f}_i\,(x_i^k)$, proceed as follows:

Initially,

$$\underline{f}_i(x_i^k) \quad = \quad 1 \text{ if } x_i^k = x_i^* ,$$

$$(3.4) \qquad\qquad = \quad 0 \text{ otherwise},$$

$$\overline{f}_i(x_i^k) \quad = \quad 0 \text{ if } x_i^k = x_{i_*},$$

$$\qquad\qquad = \quad 1 \text{ otherwise},$$

That is, if a score is the best score on a given criterion, then the lower bound on its utility is 1; otherwise, it is 0. Similarly, if a score is the worst score on a given criterion, then the upper bound on its utility is 0; otherwise, it is 1.

Having obtained $\underline{f}_i(x_i^k)$ and $\overline{f}_i(x_i^k)$, we can compute $\underline{B}^k$ and $\overline{B}^k$ using (3.1) and (3.2) respectively. The test (3.3) can then be used to determine whether certain alternatives are ruled out.

The bounds are next revised by evaluating the utility of a score:

$\underline{f}_i(x_i^k)\text{(new)} = f_i(x_i^k) = \overline{f}_i(x_i^k)\text{(new)}$. The process is repeated until a preferred alternative is identified or all scores have been evaluated.

The steps of the procedure are described below.

Step - 1 Initialization

In the first step, the w_i's are estimated and x_{i_*} and x_i^* are identified.

Step - 2 Bounding

$\underline{B}^k$ and $\overline{B}^k$ are computed using (3.1) and (3.2) respectively. Initially the $\underline{f}_i\ (x_i^k)$ and $\overline{f}_i\ (x_i^k)$ values needed in the computatic are obtained from (3.4).

Step - 3 Testing

The test in (3.3) is used to rule out certain alternatives. It is determined whether the procedure should be continued or terminated.

Step - 4 Selection

A score is selected for evaluation by the decision-maker. Some rules for selecting a score are discussed in section 4.

Step - 5 Evaluation

The selected score in step-4 is evaluated by the decision-maker and new bounds are computed in step-2.

We will now demonstrate the evaluation and bound procedure in the context of a weapon system decision problem (Table 3.1).

Table 3.1

Weapon System Decision Problem

Attributes	System X (Alternative 1)	System Y (Alternative 2)	System Z (Alternative 3)
1. Range (000 miles)	10	8	5
2. Delivery time (hr.)	5	.5	1
3. Total Yield (MT)	100	50	80
4. Accuracy (High-Low)	Average	Low	High
5. Vulnerability (High-Low)	Average	High	Low
6. Payload delivery flexibility (High-low)	High	Low	Average

Reference: Rand Corporation Memo. RM-4823-ARPA, December 1968 (MacCrimmon, K.R.)

The performance of each of the three available weapon systems is measured on six attributes. The performance or score (x_i^k) of each of the weapon systems on these attributes is known and is given in Table 3.1. For example, weapon system X has 10,000 miles range and average accuracy. The evaluation and bound procedure can be used only if attributes are valuewise independent, that is, Fishburn's marginality assumptions are satisfied. In this example we assume value-wise independence among the attributes.

We shall now apply the evaluation and bound procedure to identify a preferred weapon system.

<u>Step - 1</u> The w_i's and $\underline{f}_i(x_i^k)$ and $\overline{f}_i(x_i^k)$ for $i = 1$ to 6 and $k = 1$ to 3 are shown in Table 3.2.

<u>Step - 2</u> $\underline{B}^k$ and $\overline{B}^k$ are shown in Table 3.2.

<u>Step - 3</u> It can be seen from Table 3.2 that the lower bound on the utility of System X ($\underline{B}^1$) is greater than the upper bound on the utility of System Y ($\overline{B}^2$), and hence System Y can be dropped from further consideration.

<u>Step - 4</u> Average payload delivery flexibility is selected for evaluation. This selection is arbitrary in this example.

<u>Step - 5</u> Suppose f_6 (average) = .5 is determined by seeking judgments from the decision-maker.

<u>Step - 2</u> The revised $\underline{B}^k$ and $\overline{B}^k$ are shown in Table 3.3. The circled numbers show the new bounds obtained from step - 5.

<u>Step - 3</u> It is noted that neither System X or System Z can be ruled out.

<u>Step - 4</u> Average vulnerability is selected for evaluation.

<u>Step - 5</u> f_5(average) =.6.

Table 3.2

Initial Iteration of the Evaluation and Bound Procedure

w_i	Attributes	System X Lower Bound	System X Upper Bound	System Y Lower Bound	System Y Upper Bound	System Z Lower Bound	System Z Upper Bound
	i	$\underline{f}_i$	$\overline{f}_i$	$\underline{f}_i$	$\overline{f}_i$	$\underline{f}_i$	$\overline{f}_i$
.2	1	1	1	0	1	0	0
.1	2	0	0	1	1	0	1
.1	3	1	1	0	0	0	1
.1	4	0	1	0	0	1	1
.2	5	0	1	0	0	1	1
.3	6	1	1	0	0	0	1
		$\underline{B}^1$	$\overline{B}^1$	$\underline{B}^2$	$\overline{B}^2$	$\underline{B}^3$	$\overline{B}^3$
		.6	.9	.1	.3	.3	.8

System Y is ruled out.

Evaluate f_6(avg.) = .5 (say).

$\underline{f}_6$(new) = .5 = $\overline{f}_6$(new)

Table 3.3

First Iteration of the Evaluation and Bound Procedure

w_i	Attributes	System X Lower Bound	System X Upper Bound	System Z Lower Bound	System Z Upper Bound
	i	$\underline{f}_i$	$\overline{f}_i$	$\underline{f}_i$	$\overline{f}_i$
.2	1	1	1	0	0
.1	2	0	0	0	1
.1	3	1	1	0	1
.1	4	0	1	1	1
.2	5	0	1	1	1
.3	6	1	1	.5	.5
		$\underline{B}^1$	$\overline{B}^1$	$\underline{B}^3$	$\overline{B}^3$
		.6	.9	.45	.65

Inconclusive.

Evaluate f_5 (avg.) = .6 (say).

<u>Step - 2</u> The revised $\underline{B}^k$ and $\overline{B}^k$ are shown in Table 3.4.

<u>Step - 3</u> It can be concluded that System X is the preferred alternative, as the lower bound on the utility of System X (.72) is higher than the upper bound on the utility of System Z (.65).

Table 3.4

Second Iteration of the Evaluation and Bound Procedure

w_i	Attributes	System X		System Z	
		Lower Bound	Upper Bound	Lower Bound	Upper Bound
	i	$\underline{f}_i$	$\overline{f}_i$	$\underline{f}_i$	$\overline{f}_i$
.2	1	1	1	0	0
.1	2	0	0	0	1
.1	3	1	1	0	1
.1	4	0	1	1	1
.2	5	.6	.6	1	1
.3	6	1	1	.5	.5
		$\underline{B}^1$	$\overline{B}^1$	$\underline{B}^3$	$\overline{B}^3$
		.72	.82	.45	.65

System X is preferred.

It should be emphasized that the exact determination of the utilities of scores in step-5 is not required in our approach. For example, in Table 3.4, suppose we only know that f_5 (average) lies between .5 and .8, instead of being exactly estimated as .6. Then $\underline{f}_5^1$ = .5, and $\overline{f}_5^1$ = .8. The $\underline{B}^1$ and $\overline{B}^1$ will now be respectively .7 and .86 (Table 3.5). And, again it can be concluded that System X is preferred.

Table 3.5

Second Iteration of the Evaluation and Bound Procedure (revised)

w_i	Attributes	System X Lower Bound	System X Upper Bound	System Z Lower Bound	System Z Upper Bound
	i	$\underline{f}_i$	$\overline{f}_i$	$\underline{f}_i$	$\overline{f}_i$
.2	1	1	1	0	0
.1	2	0	0	0	1
.1	3	1	1	0	1
.1	4	0	1	1	1
.2	5	.5	.8	1	1
.3	6	1	1	.5	.5
		$\underline{B}^1$	$\overline{B}^1$	$\underline{B}^3$	$\overline{B}^3$
		.7	.86	.45	.65

System X is preferred.

3.2 Some Generalizations of the Evaluation and Bound Procedure

In this section we shall show how $\underline{B}^k$ and $\overline{B}^k$ can be obtained when the exact estimates of w_i's are not available. We also shall show that tighter bounds on the utilities of scores can be established if certain assumptions about the shape of the f_i functions are satisfied.

3.2.1 Computations of $\underline{B}^k$ and $\overline{B}^k$ with Interval Estimates on w_i

In some situations the exact estimates of w_i's may be hard to obtain. In the estimation of the w_i's, suppose the decision-maker prefers the sure reward for π' and the lottery for π''. Then, $\underline{w}_i = \pi'$ and $\overline{w}_i = \pi''$ are the lower and upper bounds, respectively, on the relative weight for the i^{th} attribute. The still finer judgments to obtain indifference may be difficult to make, but an interval range for each w_i is known, and we still require $\sum_{i=1}^{n} w_i = 1$.

To obtain $\underline{B}^k$ and $\overline{B}^k$ in step-2 of the evaluation and bound procedure (section 3.1), we can solve the following linear programs:

$$(3.5)\qquad \underline{B}^k = \text{Min} \sum_{i=1}^{n} w_i\, \underline{f}_i(x_i^k) \text{ s.t. } \sum_{i=1}^{n} w_i = 1,\ \underline{w}_i \leqq w_i \leqq \overline{w}_i$$

for all $i = 1$ to n.

$$(3.6)\qquad \overline{B}^k = \text{Max} \sum_{i=1}^{n} w_i\, \overline{f}_i(x_i^k) \text{ s.t. } \sum_{i=1}^{n} w_i = 1,\ \underline{w}_i \leqq w_i \leqq \overline{w}_i$$

for all $i = 1$ to n.

Note that $\underline{f}_i\ (x_i^k)$ and $\overline{f}_i(x_i^k)$ have been obtained by evaluating a score in step - 5 as described earlier. Thus, at each iteration we may modify the values for the constants $\underline{f}_i\ (x_i^k)$ and $\overline{f}_i\ (x_i^k)$. Fortunately, the above linear programs can be solved by inspection since (3.5) and (3.6) are simple "knapsack" problems. Below we describe the solution procedure in steps of an algorithm.

1. In (3.5) and (3.6), assign the $\underline{w}_i$ weight to all w_i, $i = 1$ to n.

2. In (3.5) successively allocate the remaining available weight $(1 - \sum_{i=1}^{n} \underline{w}_i)$ to the smallest $\underline{f}_i\ (x_i^k)$ and then to the next larger $\underline{f}_i\ (x_i^k)$ until the constraints become binding. Note that in the final solution at least (n-1) w_i's will take values $\underline{w}_i$ or $\overline{w}_i$.

3. In (3.6) successively allocate the remaining available weight $(1 - \sum_{i=1}^{n} \underline{w}_i)$ to the largest $\overline{f}_i\ (x_i^k)$ and then to the next smaller $\overline{f}_i\ (x_i^k)$ until the constraints become binding.

It can be easily seen that the solutions of (3.5) and (3.6) may have different values for the w_i's. However, the same values of w_i's should apply in the computations of both the lower and upper bounds. Based on this observation, a stronger test to rule out certain alternatives is as follows:

Solve the linear program

$$(3.7) \qquad \text{Min} \sum_{i=1}^{n} w_i \left(\underline{f}_i(x_i^k) - \overline{f}_i(x_i^\ell)\right) = Z^*$$

$$\text{s.t. } \underline{w}_i \leqq w_i \leqq \overline{w}_i \quad \text{for all } i=1 \text{ to } n, \sum_{i=1}^{n} w_i = 1.$$

If $Z^* \geqq 0$, then alternative ℓ can be excluded. The linear program in (3.7) computes the minimum possible difference between the lower and upper bounds on the utilities of the alternatives k and ℓ respectively, while satisfying the requirements on the w_i's.

(3.7) is still a simple "knapsack" problem and can be solved using the procedure described earlier. However, to test all pairs of alternatives, N(N-1) linear programs need to be solved as compared with only 2N linear programs that need to be solved for computing the individual alternative bounds ($\underline{B}^k$ and $\overline{B}^k$) as in (3.5) and (3.6). A good strategy could be to first compute $\underline{B}^k$ and $\overline{B}^k$ for k = 1 to N using (3.5) and (3.6) respectively, and then do the additional test in (3.7) for the pairs with small $(\overline{B}^\ell - \underline{B}^k)$. Of course if this approach is implemented on a time sharing computer system, the additional burden of solving the $N(N-1)$ versions of (3.7) may not be significant.

3.2.2 Computation of $\underline{f}_i(x_i^k)$ and $\overline{f}_i(x_i^k)$ under Certain Assumptions on the Shape of f_i

So far, we have made no assumptions about the shape of the f_i functions. Thus, whenever the decision-maker evaluates a score x_i^k, only $\underline{f}_i(x_i^k)$ and $\overline{f}_i(x_i^k)$ are revised. If the f_i are monotonic, the lower and upper bounds of several other scores x_i^t, $t \neq k$, can be revised simultaneously. Suppose, for some i, $f_i(x_i)$ is either monotonic non-increasing or monotonic non-decreasing in x_i, and the utility of a score x_i^k is evaluated ($\underline{f}_i(x_i^k)$ are determined where $\underline{f}_i(x_i^k) = \overline{f}_i(x_i^k)$ if the decision-maker gives an indifference judgment). Then, the bounds for all alternatives ℓ for which $x_i^k \geqq x_i^\ell$, and the alternatives t for which $x_i^k \leqq x_i^t$ can be revised as follows:

$$(3.8)\qquad \left.\begin{aligned} \underline{f}_i(x_i^\ell) &= \underline{f}_i(x_i^k), \\ \overline{f}_i(x_i^t) &= \overline{f}_i(x_i^k) \end{aligned}\right\} \text{ for non-increasing } f_i.$$

and

$$(3.9)\qquad \left.\begin{aligned} \underline{f}_i(x_i^t) &= \underline{f}_i(x_i^k), \\ \overline{f}_i(x_i^\ell) &= \overline{f}(x_i^k) \end{aligned}\right\} \text{ for non-decreasing } f_i.$$

Similarly, a prior knowledge about the concavity or the convexity of the f_i functions can b used to obtain tighter bounds. For example, suppose f_i is concave and monotonic and $f_i(x_i^\ell)$ and $f_i(x_i^t)$ have been obtained, then $\underline{f}_i(x_i^k)$, $x_i^\ell \leq x_i^k \leq$ can be obtained by linear extrapolation between the two values x_i^ℓ and x_i^t.

$$\underline{f}_i(x_i^k) = f_i(x_i^\ell) + \frac{(x_i^k - x_i^\ell).\ (f_i(x_i^t) - f_i(x_i^\ell))}{(x_i^t - x_i^\ell)}$$

If f_i is convex and monotonic then $\overline{f}_i(x_i^k)$ can be similarly obtained.

$$\overline{f}_i(x_i^k) = f_i(x_i^\ell) + \frac{(x_i^k - x_i^\ell).\ (f_i(x_i^t) - f_i(x_i^\ell))}{(x_i^t - x_i^\ell)}$$

Note that we cannot obtain both the lower and the upper bounds with these properties at the same time.

3.3 Evaluation and Bound Procedure and Multiple Events

The generalization of the evaluation and bound procedure for situations with multiple events should be obvious. In step-2 of the evaluation and bound procedure the lower and upper bounds on the utilities of an alternative are obtained under each of the possible events. These are then weighted with the respective probabilities to obtain lower and upper bounds on the expected utilities of the alternatives. Specifically, let $\underline{B}_j^k$ and $\overline{B}_j^k$ be the lower and upper bounds respectively of alternative k under event j. Then, a lower bound on the expected utility of alternative k $(E(\underline{B}^k))$ is $\sum_{j=1}^{m} p_j^k \cdot \underline{B}_j^k$. Similarly, the upper bound on the expected utility of alternative $k(E(\overline{B}^k))$ is $\sum_{j=1}^{m} p_j^k \cdot \overline{B}_j^k$. In step-3 of the evaluation and bound procedure these bounds are used to eliminate some alternatives.

4. Experimental Results

An extensive experimental study has been undertaken to provide empirical evidence on the evaluation and bound approach with the following objectives. First, it was desired to determine the average savings in information requirement that can be realized by using the Evaluation and Bound procedure. Second various rules were tested for selecting a score for evaluation. The second objective is relevant particularly for implementing the procedure on an interactive computer.

A 4 x 4 factorial design of experiments was adopted by varying both the number of attributes and the number of alternatives from three to twelve in steps of three. For each of the sixteen problem sizes, three distinct experiments were conducted. In first experiment, the relative weights (w_i's) for each criterion were randomly generated for each of the 40 randomly generated problems. In the second experiment, the w_i's were assumed to be uniform (roughly same value for each criterion) and were held constant for each of the 30 randomly generated problems. In the third experiment, the w_i's were assumed to be skewed (some attributes have considerably higher value that others) and were held constant for each of the 30 randomly generated problems.

The results showed that the percentage savings in the number of judgments required from the decision-maker is substantial in all three situations (see Sarin [13]) for details). The highest savings are realized when the distribution of relative weights is skewed and the lowest when it is uniform. The variation in the number of judgments required is least when the weights are uniform and the greatest when the weights are skewed.

Three rules for selecting a score for evaluation by the decision-maker were examined and compared experimentally. In rule 1, the unevaluated attribute with the highest relative weight w_i is scanned. An arbitrary score x_i^k which has not yet been evaluated is selected for evaluation. Once all of the scores, the x_i^k's, on this attribute have been evaluated, the process is repeated for the unevaluated attributes with the next lower relative weights. Scanning terminates when either a preferred alternative is identified, or all scores have been evaluated. In rule 2, the weighted difference D_i^k between the upper and lower bounds on the utilities of scores is computed, where $D_i^k = w_i \cdot (\bar{f}_i(x_i^k) - \underline{f}_i(x_i^k))$. The unevaluated score with the highest D_i^k is evaluated next. Rule 3 is based on the linear extrapolations of the functions, the f_i's, to determine which score would yield the maximum tightening on the lower and upper bounds on the utilities of the alternatives.

The evaluation and bound procedure was applied to several randomly generated problems to determine the relative number of evaluations required for the identification of a preferred alternative when each of the three selection rules are used. It can be concluded from the experiments that rule 2 is superior to the rule 1. However, there is no significant difference between rule 2 and rule 3. Rule 3 requires considerably more computations in selecting a score than rule 2. We therefore recommend that rule 2 be used in selecting a score for evaluation by the decision-maker.

5. Summary and Some Extensions

An evaluation and bound procedure for the selection of multi-attributed alternatives has been developed. The procedure solicits additional information from the decision-maker sequentially, through an interactive process, until a final choice is determined. This sequential evaluation process involves two phases. In one phase the analyst (or computer) calculates bounds and performs tests to exclude some alternatives. In the other phase, the decision-maker evaluates an outcome and supplies his evaluation to the analyst. Based on this information, the analyst repeats phase one. The procedure terminates when a preferred alternative is identified. The progress of the sequential evaluation process is guided by a scheme (rule) for selecting a criterion score for evaluation by the decision-maker. It was demonstrated that this procedure reduces considerably the number of judgments needed from the decision-maker in identifying a preferred alternative.

The major research direction is the development of a computerized man-machine interactive system, and experimentation with subjects on some real life problems. It would also be desirable to incorporate sensitivity analysis into the system, so that the decision-maker is provided with the flexibility to revise some of his evaluations. The system should also include the consistency checks to detect and query the decision-maker about inconsistent judgments.

The contribution of the research reported here is in reducing the information burden on the decision-maker, and thus providing practical assistance to him. It is hoped that this research would bridge the gap between the theoretical advancements in the multi-attributed utility theory and its applications to real life decision situations. Potential applications of this research can be in criminal justice project evaluation, research and development project evaluation, evaluation of certain class of education projects and some projects in health systems. The procedure developed here is equally applicable for the selection of plans, strategies, actions, research proposals, new products, and for many other decision situations in both private and public sectors.

REFERENCES

1. Aumann, R.J., and Kruskal, J.B., "The Coefficients in an Allocation Problem," Naval Research Logistics Quarterly, Vol 5, 1958, pp. 111-123.

2. ________. and Kruskal, J.B., "Assigning Quantitative Values to Qualitative Factors in the Naval Electronics Problem," Naval Research Logistics Quarterly, Vol 6, 1959, pp. 1-16.

3. Debreu, G. "Representation of a Preference Ordering by a Numerical Function," in R.M. Thrall, C.H. Coombs and R.L. Davis (Eds.), Decision Processes, John Wiley and Sons, New York, 1954.

4. Dyer, James S. "A Procedure for Selecting Educational Goal Areas for Emphasis," Operations Research, Vol 21, May-June, 1973, pp.835-845.

5. ________. and Lientz, Bennet, P. and Sarin, Rakesh K. "The Evaluation of Alternative Projects in Criminal Justice Planning," Paper presented at ORSA/TIMS Conference, Puerto Rico, Oct. 16-18, 1974.

6. Eckenrode, R.T. "Weighting Multiple Criteria," Management Science, Vol 12, 1965, pp. 180-192.

7. Fishburn, P.C. "Independence in Utility Theory with Whole Product Sets," Operations Research, Vol 13, 1965, pp. 28-45.

8. Fishburn, P.C. "Independence, Trade-Offs, and Transformations in Bivariate Utility Functions," Management Science, Vol 11, 1965, pp 792-801.

9. ____________. Utility Theory for Decision Making, Wiley: New York, 1970.

10. Keeney, R.L., "Multiplicative Utility Functions," Operations Research, Vol 22, No 1, 1974, pp. 22-34.

11. MacCrimmon, K.R., "Decision-making Among Multiple-Attribute Alternatives: A Survey and Consolidated Approach," Rand Corporation Memo. RM-4823-ARPA, Dec. 1968

12. Raiffa, Howard, "Preferences for Multiattributed Alternatives," RM-5868-DOT/RC. The Rand Corporation, April, 1969.

13. Sarin, R.K., "Interactive Approaches for the Evalutation of Multi-attributed Alternatives," Unpublished doctoral dissertation at University of California, Los Angeles, June, 1975.

14. Stillson, P., "A Method for Defect Evaluation" Industrial Quality Control, Vol 11, 1954, pp. 8-12.

15. von Neumann, T. and Morgenstern, O., Theory of Games and Economic Behavior, 2nd Ed., Princeton University Press, Princeton, New Jersey, 1947.