DYNAMIC OPTIMIZATION WITH INTEGRAL STATE EQUATIONS

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ABSTRACT

In this paper we obtain necessary and sufficient conditions for optimality in dynamic optimization problems governed by integral state equatiions.

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My interest in this problem originated in very useful discussions that I had with Jim Jordan and Ramon Marimon back in 1986. Subsequent discussions with Dipankar Dasgupta, (Academician) V.I. Tsurkov and Girja Sharan have stimulated my interest in this problem.

Introduction : In Kamien and Muller (2), the prototype optimal control problem was generalized to incluce dynamic optimization problems with integral state equations. A formal solution of the problem based on the maximum principle is available in Bakke (1) and Vinokurov (5). Kamien and Muller (2) cite extensive references of the application of the above mentioned model to problems of economics and management. However, in the last mentioned paper what was provided was a heuristic argument in support of the necessary and sufficient conditions.

Our purpose in this paper is to present an easily accessible development of alternative necessary and sufficient conditions based on the methods of dynamic programming.

2. Model : Formally, the general problem is to maximize

$$\int_a^b I\,(t,x(t),u(t))dt + F(x_1,b) \tag{1}$$

Subject to ;

$$x(t) = \int_a^t f(g(t),x(s),u(s),s)ds + x(a) \tag{2}$$

$$x(b) = x_1 \tag{3}$$

$$u(t) \in \Omega \quad \forall \quad t \in [a,b] \tag{4}$$

$$(x_1,b) \in T \subseteq \mathbb{R}^{n+1} \tag{5}$$

where $x(t) \in \mathbb{R}^n \ \forall \ t \in [a,b]$, $\Omega \subseteq \mathbb{R}^r$. Further we require that the state trajectory $\{x(t)\} = \{x(t) \in \mathbb{R}^n / a \leq t \leq b\}$ is a continuously differentiable function of time, starting at the initial state $x(a)$ which is given and ending at the terminal state $x(b)=x_1$, which must be determined. At each point in time 't' the decisions to be made are characterized by the vector $u(t) \in \Omega$ called the control variable. It is required that the control trajectory $\{u(t)\} = \{u(t) \in \Omega / a \leq t \leq b\}$ is a piecewise continuous function of time. Equation (1) is the equation of motion and T the terminal surface. The function $I : [a,b] \times \mathbb{R}^n \times \Omega \to \mathbb{R}$ is called the intermediate function and the function $F : \mathbb{R}^n \times [a,b] \to \mathbb{R}$ is called the terminal function. We assume that I, F and Ω are such that both I and F are continuously differentiable functions of their respective arguments. In addition we assume that the function $g: [a,b] \to \mathbb{R}$ is continuously differentiable and the function 'f' admits continuous partial derivatives with respect 'g'. (Kamien and Muller(2) consider the special case where $g(t) = t \; \forall \; t \in [a,b]$). we also assume that f satisfies all the conditions required to generate a unique state trajectory for each admissible control trajectory on $[a,b]$.

Assuming a solution exists for the general control problem(1) - (5) let :

$$J^* (x, \tau) \qquad (6)$$

be the optimal performance function, the maximized value of the objective functional for the problem starting at the initial state x at time 't'.

Suppose $J^* : \mathbb{R}^n \times \mathbb{R}_+ \to \mathbb{R}$ is a single valued and continuously differentiable function of the n+1 variables. Then the fundamental recurrence relation of dynamic programming, called Bellman's equation, is summarized in the following :

$$-\frac{\partial J^*}{\partial t}(x, \tau) = \max_{u(t) \in \Omega} \left[I(x, u(\tau), \tau) + \frac{\partial J^*}{\partial x}(x, \tau)\, \dot{x}(\tau) \right] \qquad (7)$$

$$J^*(x_1, b) = F(x_1, b). \qquad (8)$$

(7) and (8) are valid in our context owing to the form of the objective function and the priciple of optimality. The following theorem summarizes a main result of our analysis.

Theorem 1 : If $\{x^*(t) : a \le t \le b\}$ be an optimal state trajectory and $\{u^*(t) : a \le t \le b\}$ be an optimal control trajectory which together solves the problem given by (1) - (5) and if J^* is the optimal performance function for the problem, then

$$-\frac{\partial J^*}{\partial t}(x^*(\tau), \tau) = \max_{u(\tau) \in \Omega} \Big[I(x^*(\tau), u(\tau), \tau)$$
$$+ \frac{\partial J^*}{\partial x}(x^*(\tau), \tau) \left[f(g(\tau), x^*(\tau), u(\tau), \tau) + \left[\int_a^{\tau} f_g(g(\tau), x^*(s), u(s), s)\, ds \right] \frac{dg(\tau)}{dt} \right] \Big]$$

$$\left[I(x^*(\tau),u^*(\tau),\tau) + \frac{\partial J^*}{\partial x}(x^*(\tau),\tau)\left\{f(g(\tau),x^*(\tau),u^*(\tau),\tau) + \left[\int_a^{\tau} f_g(g(\tau),x^*(s),u^*(s)ds\right]\frac{dg(\tau)}{dt}\right\}\right]$$

<u>Proof</u> : The proof is analogous to the proof of Bellman's recurrence equation in the standard case (as for instance in Kamien and Schwartz(3)) once we observe that (2) can be expressed as follows : Write,

$$x(t,g(t)) = \int_a^t f(g(t),x(s),u(s),s)ds + x(a)$$

Then
$$x(\tau) = \frac{\partial x}{\partial t}(\tau,g(\tau)) + \frac{\partial x}{\partial g}(\tau,g(\tau))\frac{dg}{dt}(\tau)$$

$$= f(g(\tau),x(\tau),u(\tau),\tau) + \left[\int_a^{\tau} f_g(g(\tau),x(s),u(s),s)ds\right]\frac{dg(\tau)}{dt} \qquad (9)$$

Subsituting this in Bellman's recurrence relation (7) and observing that along an optimal state trajectory $\{x^*(t), a \leq t \leq b\}$, the maximum on the R.H.S. of (7) is attained at $u^*(t)$ for each $t \in [a,b]$ we obtain the result mentioned in the theorem.

Q.E.D.

The converse of this theorem is proven below :

<u>Theorem 2</u> : If a continuously differentiable function $J^* : \mathbb{R}^n \times \mathbb{R}_+ \to \mathbb{R}$ can be found that satisfies the equation

$$-\frac{\partial J^*}{\partial t}(x(\tau),\tau)) = \max_{u(\tau)\in\Omega}\left[I(x(\tau),u(\tau)\tau) + \frac{\partial J^*}{\partial x}(x(\tau),\tau)\left\{f(g(\tau),x(\tau),u(\tau),\tau) + \left[\int_a^{\tau} f_g(g(\tau),x(s),u(s),s)ds\right]\frac{dg(\tau)}{dt}\right\}\right]$$

Subject to the boundary condition (8) then it generates the optimal control trajectory to problem (1) - (5) through the static (pointwise) maximization problem defined by the RHS of the above equation.

<u>Proof</u> : Let $\{x(t), a \leq t \leq b\}$ be any other state trajectory associated to the control $\{u(t), a \leq t \leq b\}$ satisfying (2) - (5) and (8). Then

$$I(x(\tau),u(\tau),\tau) + \frac{\partial J^*}{\partial x}(x(\tau),\tau)\Big\{f(g(\tau),x(\tau),u(\tau),\tau)+\Big[\int_a {}^{t} g \ldots \quad dt \quad \Big]$$

$$+\frac{\partial J^*}{\partial t}(x(\tau),\tau) \leq I(x^*(\tau),u^*(\tau),\tau)+\frac{\partial J^*}{\partial x}(x^*(\tau),\tau)\Big\{f(g(\tau),x^*(\tau),u^*(\tau),\tau)$$

$$+\left[\int_a^{\tau} f_g(g(\tau),x^*(s),u^*(s),s)\,ds\right]\frac{dg(\tau)}{dt}\Big\}+\frac{\partial J^*}{\partial t}(x^*(\tau),\tau) = 0 \;\forall\; \tau \in [a,b].$$

Integrating the left hand side from a to be and the right hand side from a to b^* we get ;

$$\int_a^b I(x(t),u(t),t)dt+J^*(x(b),b)-J^*(x_0,a) \leq \int_a^{b^*} I(x^*(t),u^*(t),t)dt+J^*(x^*(b^*),b^*)-J^*(x_0,a)$$

Elimination of $J^*(x_0,a)$ yields

$$\int_a^b I(x(t),u(t),t)dt + F(x(b),b) \leq \int_a^{b^*} I(x^*(t),u^*(t),t)dt + F(x^*(b^*),b^*),$$

from which it readily follows that u^* is the optimal control.

Q.E.D.

3. The Maximum Principle :

In this sub-section our starting point will be the Bellman equation (7) with (8), under the additional assumption that J is twice continuously differentiable.

Let,

$$\tilde{H}(t,x,u) = \frac{\partial J^*}{\partial x}(x,t)f(g(t),x,u,t) + I(t,x,u) \qquad (10)$$

In terms of (7) and (9) equation (10) can be written as

$$-\frac{\partial J^*}{\partial t}(x,t) = \max_{u\in\Omega} \tilde{H}(t,x,u)+\frac{\partial J^*}{\partial t}(x,t)\left[\int_a^t f_g(g(t),x(s),u(s),s)ds\right]\frac{dg(t)}{dt} \qquad (11)$$

Being consistent with our earlier convention, the maximizing u will be denoted by u^*. Then

$$\tilde{H}(t,x,u^*)+\frac{\partial J^*}{\partial t}(x,t) + \frac{\partial J^*}{\partial x}(x,t)\left[\int_a^t f_g(g(t),x(s),u(s),s)ds\right]\cdot\frac{dg(t)}{dt} = 0 \qquad (12)$$

This is an identity in x and the partial derivatives of this function with respect to x are also zero. Thus :

$$\frac{\partial^2 J^*}{\partial x^2}(x,t) f(g(t),x,u^*,t) + \frac{\partial J^*}{\partial x}(x,t)\frac{\partial f}{\partial x}(g(t),x,u^*,t) + \frac{\partial I}{\partial x}(t,x,u)$$

$$+\frac{\partial^2 J^*}{\partial x \partial t}(x,t) + \frac{\partial^2 J^*}{\partial x^2}(x,t) \cdot \left[\int_a^t f_g(g(t),x(s),u(s),s)ds\right] \cdot \frac{dg(t)}{dt} = 0 \qquad (13)$$

Taking the total derivative of $\frac{\partial J^*}{\partial x}(x,t)$ with respect to 't' we get :

$$\frac{\partial^2 J^*}{\partial x^2}(x,t) . \dot{x}(t) + \frac{\partial^2 J^*}{\partial t \partial x}(x,t) = \frac{d}{dt}\left(\frac{\partial J^*}{\partial x}(x,t)\right) .$$

Interchanging the orders of second partial derivatives of J^* (which requires J^* to be twice continuously differentiable) we get

$$\frac{d}{dt}\left(\frac{\partial J^*}{\partial x}(x,t)\right) + \frac{\partial J^*}{\partial x}(x,t) \frac{\partial f}{\partial x}(g(t),x,u^*,t) + \frac{\partial I}{\partial x}(t,x,u) = 0 \qquad (14)$$

This we obtain from (13).

By introducing the so called <u>costate vector</u>, $p'(t) = \frac{\partial J^*}{\partial x}(x^*,t)$, where x^* denotes the state trajectory corresponding to u^*, equation (14) can be written as

$$\frac{d}{dt} p'(t) = - \frac{\partial}{\partial x}\left[I(t,x^*,u^*)+p'(t)f(g(t),x^*,u^*,t)\right] \equiv - \frac{\partial}{\partial x} H(t,p,x^*,u^*) \qquad (15)$$

where H is defined by

$$H(t,p,x,u) \equiv I(t,x,u) + p'(t)f(g(t),x,u,t) \qquad (16)$$

The boundary condition for $p(t)$ is determined from

$$p'(b) = \frac{\partial J^*}{\partial x}(x^*,b) = \frac{\partial F}{\partial x}(x^*,b) \text{ on } (x,b) \in T \subseteq \mathbb{R}^{n+1} \qquad (17)$$

In conclusion, we have arrived at the following necessary condition for the optimal control $u^*(.)$:

<u>Theorem 3</u> : Under the assumption that the value function $J^*(t,x)$ is twice continuously differentiable, the optimal control $\{u^*(t) : a \leq t \leq b\}$ and corresponding state trajectory $\{x^*(t) : a \leq t \leq b\}$ must satisfy the following so called <u>canonical equations</u> :

$$
\left.
\begin{aligned}
x^*(t) &= \int_a^t f(g(t), x^*(s), u^*(s), s)\,ds \\
\dot{p}'(t) &= -\frac{\partial H(t,p,x^*,u^*)}{\partial x} \\
p'(b) &= \frac{\partial F(x^*,b)}{\partial x} \quad \text{on } (x,b) \in T \subseteq \mathbb{R}^{n+1} \\
H(t,p,x,u) &\equiv I(t,x,u) + p'(t) f(g(t),x,u,t) \\
u^*(t) &= \arg\max_{u \in \Omega} H(t,p,x^*,u)
\end{aligned}
\right\} \qquad (16)
$$

<u>Remark 1</u> : A particular class of dynamic optimization problems with **<u>fixed terminal condition</u>**, i.e. with the terminal time 'b' and the terminal state x(b) prescribed, requires omitting the boundary condition p'(b) with the new condition $x(b) = x_1$ (given).

<u>Remark 2</u> : With $g(t) = t \; \forall \; t \geq 0$, we are in the framework analysed by Kamien and Muller (2). A general 'g', does not in any way add to greater generality; it only helps to highlight the dependence of x on a parameter in addition to 't'.

<u>Remark 3</u> : Our necessary conditions for optimality are different from the ones obtained earlier. However, they are easier to conceptualize and solve than the ones available in the literature. The costate variables in our framework are the shadow prices of the state variables (an interpretation consistent with the interpretation of costate variables in optimal control theory governed by differential equations) and have values which are different from the values of the Lagrange Multipliers used by Kamien and Muller (2). In the special case where $f_g(.) \equiv 0$ (i.e. where f does not depend on any parameter) both formulations are identical to the special theory of differential equations governed by differential equations.

4. <u>Sufficient Conditions</u> :

<u>Theorem 1</u> : Let $\{u^*(t): a \leq t \leq b\}$ and the corresponding $\{x^*(t): a \leq t \leq b\}$, $\{p(t): a \leq t \leq b\}$ satisfy (16). Then $\{u^*(t): a \leq t \leq b\}$ is an optimal control if

$$H^o(t,p,x) = \underset{u \in \Omega}{\text{maximum}} \left[H(t,p,x,u) + p'(t) \left[\int_a^t f_g(g(t),x(s),u(s),s)ds \right] \cdot \frac{dg(t)}{dt} \right] \quad (19)$$

called the <u>derived Hamiltonian</u>, is concave in x for each t and $F(x_1,b)$ is concave in x_1.

<u>Proof</u> : The proof is adapted from Sethi and Thompson (4). By definition

$$H(t,p(t),x(t),u(t)) \le H^o(t,p(t),x(t)) + p'(t)\left[\int_a^t f_g(g(t),x(s),u(s),s)ds\right] \frac{dg(t)}{dt} \quad (20)$$

From concavity of H^o,

$$H^o\left[t,p(t),x(t)\right] \le H^o\left[t,p(t),x^*(t)\right] + \frac{\partial}{\partial x} H^o\left[t,p(t),x^*(t)\right]\left[x(t)-x^*(t)\right]; \quad (21)$$

Thus from (18), (19), (20) and (21)

$$\begin{aligned} &H\left[t,p(t),x(t),u(t)\right] + p'(t)\left[\int_a^t f_g(g(t),x(s),u(s),s)ds\right] \cdot \frac{dg(t)}{dt} \\ &\le H(t,p(t),x^*(t),u^*(t)) + p'(t)\left[\int_a^t f_g(g(t),x^*(s),u^*(s),s)ds\right] \cdot \frac{dg(t)}{dt} \\ &\quad + \frac{\partial}{\partial x} H^o\left[t,p(t),x^*(t)\right]\left[x(t)-x^*(t)\right] \end{aligned} \quad (22)$$

By definition of H and the adjoint equation

$$\begin{aligned} &I(t,x(t),u(t)) + p'(t)f(g(t),x(t),u(t),t) + p'(t)\left[\int_a^t f_g(g(t),x(s),u(s),s)ds\right] \frac{dg(t)}{dt} \\ &\le I(t,x^*(t),u^*(t)) + p'(t)f(g(t),x^*(t),u^*(t),t) + p'(t)\left[\int_a^t f_g(g(t),x^*(s),u^*(s),s)ds\right] \frac{dg(t)}{dt} \\ &\quad - \dot{p}'(t)\left[x(t) - x^*(t)\right] \end{aligned} \quad (23)$$

$$\therefore\ I(t,x^*(t),u^*(t)) - I(t,x(t),u(t)) \ge \dot{p}'(t)\left[x(t)-x^*(t)\right] + p'(t)\left[\dot{x}(t)-\dot{x}^*(t)\right] \quad (24)$$

Furthermore, since $F(x_1,b)$ is a concave function, we have

$$F(x(b),b) \le F(x^*(b),b) + \frac{\partial F}{\partial x}(x^*(b),b)\left[x(b) - x^*(b)\right]$$

$$\text{or } F(x^*(b),b) - F(x(b),b) + \frac{\partial F}{\partial x}(x^*(b),b)\left[x(b) - x^*(b)\right] \ge 0$$

Integrating both sides of (24) from a to b and adding (26) we have

$$\begin{aligned} &\int_a^b I(t,x^*(t),u^*(t))dt + F(x^*(b),b) - \int_a^b I(t,x(t),u(t))dt - F(x(b),b) \\ &\quad + \frac{\partial F}{\partial x}(x^*(b),b)\left[x(b)-x^*(b)\right] \ge p'(b)\left[x(b) - x^*(b)\right] - p(a)\left[x(a)-x^*(a)\right] \end{aligned} \quad (27)$$

Since $x(a)=x^*(a)=x_0$ and since $p'(b)=\frac{\partial F(x^*(b),b)}{\partial x}$, we get the desired result.

Q.E.D.

Remark 1 : The above theorem gives conditions under which the necessary conditions for optimality are also sufficient. Differentiability of H° with respect to 'x' is implicitly required.

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