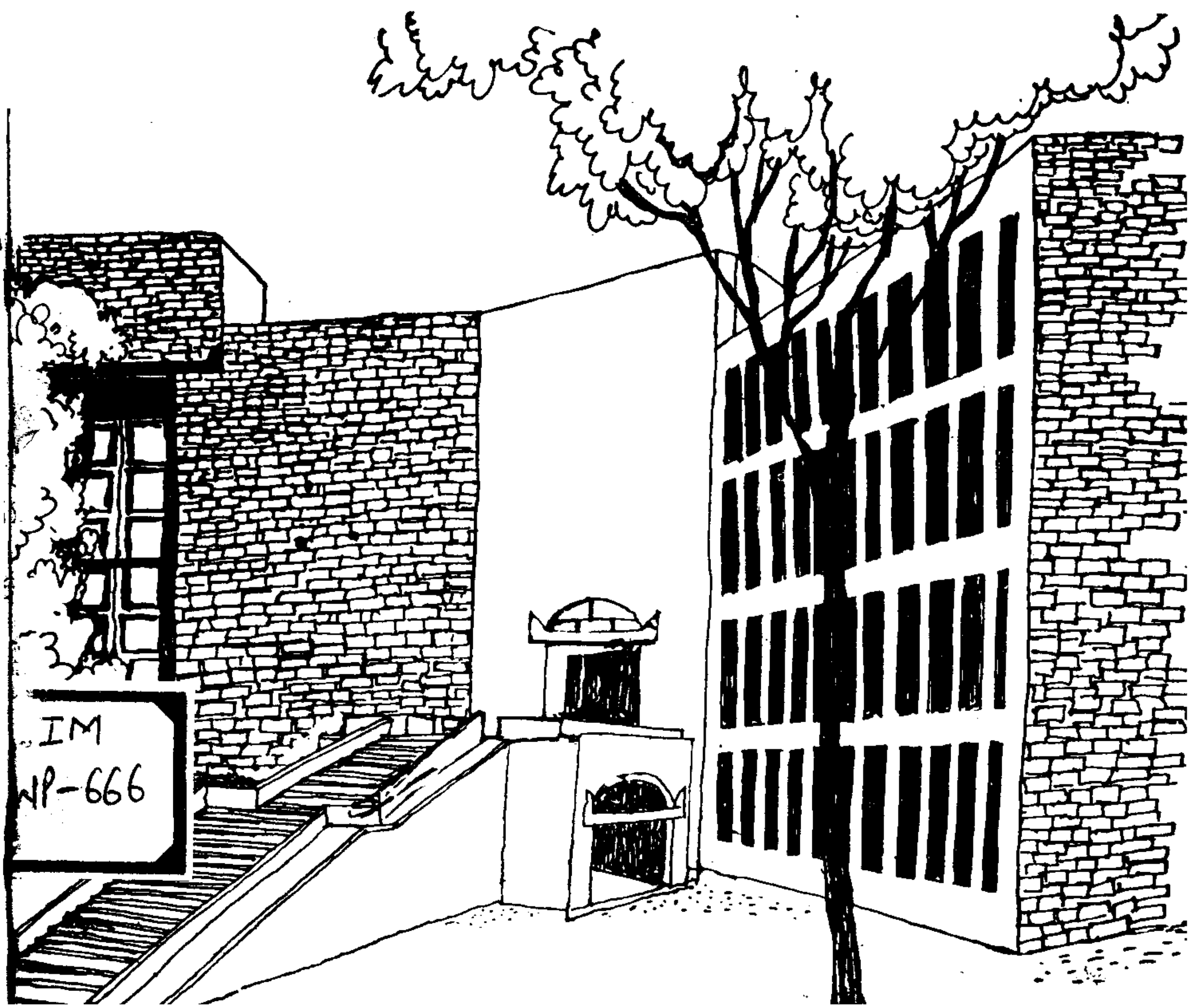
विद्याविनियोगाद्विकासः

IIM
AHMEDABAD

W. P. 666

Working Paper



G. D. P. M.
W. P. 666
APRIL 1987
8103
C/L

QUANTITY DISCOUNT PRICING MODEL : AN EXACT FORMULATION AND ANALYSIS

By

G. Srinivasan
&
V. Venkata Rao

W P No. 666
April, 1987

INDIAN INSTITUTE OF MANAGEMENT
AHMEDABAD-380015
INDIA

Abstract

In this paper we formulate and analyze the quantity discount pricing problem without the approximation that was earlier used by Monahan [3] and Lee and Rosenblatt [2]. Our exact analysis throws light on some important conceptual implications of the above approximation. The exact formulation also enables us to discuss the discounting problem separately from the view-points of seller, buyer and the total system. Specifically, we show that the optimal policy from the buyer's view-point should be the same as that from the system view-point; and that the optimal policies of the buyer and the seller need not be the same. In addition, we present procedures for computing the optimal policies for the above three cases.

Quantity Discount Pricing Model : An Exact Formulation and Analysis

1. Introduction

Recently there has been a revival of research on the determination of economic order quantity for the case of quantity discounts. Monahan's paper [3] is chiefly responsible for renewing the researchers interest in this problem, which in the past was analysed only from the buyer's view-point [1]. Unlike in the traditional approach, Monahan did not assume as given the lot quantity for which discount applies; instead, the unit price discount and the corresponding lot quantities are to be determined by the seller. Monahan confined his study to the case where (1) the supplier supplies to only one buyer, and (2) the supplier incurs a setup whenever the buyer places an order. Later, Lee and Rosenblatt [2] generalised the results of Monahan by dealing with the case where one setup of the seller can cover more than one order of the buyer.

The results of both Monahan [3] and Lee and Rosenblatt [2] were, as the authors themselves stated, approximate : they ignored in their analysis the change in unit inventory holding cost due to discounting. The objective of this paper is to study the discounting problem without the above approximation. The exact formulation of this problem enables us to do the following: (1) bring to light some important conceptual implications of the approximation used by Monahan, and Lee and Rosenblatt, (2) show that for a given lot quantity, there is a range of discount prices, each value within which is acceptable to both the buyer and the seller, and (3) propose optimal strategies for the buyer, seller, and the total system.

In section 2 of this paper we develop the model, in section 3 we analyse the implications of the approximation used by Monahan, and Lee and Rosenblatt, and, finaly, in section 4 we discuss the optimal strategies separately for the buyer, seller, and the total system.

2. The Model

The model that we present in the rest of the paper is based on the following assumptions :

1. The buyer is operating under the standard EOQ conditions : his demand is continuous, deterministic, and follows a constant rate; lead time of procurement is zero; and the planning horizon is infinite.

2. The seller caters to a single buyer; and, his setup time for production is zero, and production rate infinite. These assumptions imply that the optimal lot size produced by the seller in a single setup has to be an integer multiple of the optimal order size of the buyer [2].

A part of our notation is shown in Table-1.

Table 1 : Summary of notation

Symbol	Meaning	Units of measurement
D	Annual demand faced by the buyer	item units per year
C_b	Unit purchase cost of the buyer	money units per item
H_b	Inventory carrying charge factor of the buyer	money units per money unit per year
A_b	Ordering cost per order of the buyer	money units
C_s	Unit production cost of the seller	money units per item
H_s	Inventory carrying charge factor of the seller	money units per money-unit per year
A_s	Setup cost per setup of the seller	money units
Q_b	Optimal order size (EOQ) of the buyer	item units
Q_s	Optimal batch size of the seller	item units
N	The integer multiple that determines Q_s, for a given Q_b, such that $Q_s = NQ_b$	
TC	The total annual cost (purchase + ordering + inventory) incurred by the buyer	money units per year
TR	The total annual profit earned by the seller	money units per year

Under the above assumptions and notation, the optimal order size Q_b, of the buyer is given by

$$Q_b = ((2\, D A_b)/(H_b C_b))^{0.5} \qquad \text{-------} \quad (1)$$

The total annual cost TC to the buyer is given by

$$TC = (A_b D/Q_b) + (H_b C_b Q_b/2) + (C_b D) \qquad \text{-------} \quad (2)$$

According to assumption 2 above, the seller's optimal lot size, Q_s, of production has to be an integer muiltiple of the lot size of procurement Q_b of the buyer. Let $Q_s = NQ_b$ where N is an integer, greater than or equal to 1. N can be interpreted as the number of orders of the buyer covered by one setup of the seller. The total annual profit, TR_s, of the seller is given by

$$TR_s = (C_b D) - (A_s D/(Q_b N)) - ((N-1)\, Q_b H_s C_s/2) \qquad \text{--------} \quad (3)$$

It can be easily shown that to maximize TR_s, N has to be the largest integer that satisfies the inequality

$$N\,(N-1) \leq 2A_s D/(Q_b^2 H_s C_s) \qquad \text{--------} \quad (4)$$

A quantity-discount scheme consists of the seller offering to sell each unit at a reduced price C_d, $C_d < C_b$, provided the buyer agrees to increase his lot size from Q_b to Q_d, $Q_d > Q_b$. The reduced price applies to all units sold and hence it is called all-unit discount.

Under a given discount scheme, that is, for given values of C_d and Q_d, let the total annual profit of seller be TR_d and the total annual cost of the buyer TC_d. TC_d and TR_d are given by

$$TC_d = (A_b D/Q_d) + (H_b C_d Q_d/2) + (C_d D) \qquad \text{-------- (5)}$$

$$TR_d = (C_d D) - (A_s D/(Q_d N_d)) - ((N_d - 1) Q_d H_s C_s/2) \qquad \text{-------- (6)}$$

where N_d is the optimal number of the buyer's orders to be covered by one setup of the seller, under the discount conditions. N_d is, as before, the largest integer that satisfies the inequality

$$N_d (N_d - 1) \leq (2A_s D/(Q_d^2 H_s C_s)) \qquad \text{-------- (7)}$$

According to the above expressions, the effects of discounting for the buyer are as follows:

1.The total amount spent for purchasing decreases, because $C_d < C_b$

2. The annual number of orders and hence the annual ordering cost decreases, because $Q_d > Q_b$

3. The annual inventory carrying cost may increase, decrease, or stay the same as before depending on whether

$$C_d Q_d > C_b Q_b \text{ or } C_d Q_d \leq C_b Q_b .$$

Thus the net effect of changes in all these costs may or may not result in a saving for the buyer. Let us denote by S_b the savings in annual cost of the buyer due to a discounting scheme. S_b is given by

$$S_b = TC_b - TC_d$$

$$= A_b D [(1/Q_b)-(1/Q_d)]+D (C_b - C_d)-(H_b/2)(Q_d C_d - Q_b C_b) \quad \text{--------(8)}$$

Similarly, the effects of discounting on the seller's costs and revenue are as follows:

1. The total annual revenue decreases because $C_d < C_b$

2. The total annual cost of setups and inventory may increase, decrease or stay the same depending on whether

$$N_d Q_d > NQ_b \text{ or } N_d Q_d \leq NQ_b .$$

Thus, even for the seller, whether a given discount scheme results in an increase in profit depends on the values of the parameters under consideration. Denoting by S_s the increase in seller's annual profit due to a discounting scheme, we get

$$S_s = TR - TR_d = A_s D [(1/(NQ_b)) - (1/(N_d Q_d))]$$],

$$- H_s C_s [(Q_d (N - 1)/2) - (Q_b (N-1)/2)] - D (C_b - C_d) \qquad \text{--------} \quad (9)$$

Therefore, a proposed discounting scheme is not acceptable to the seller if $S_s < 0$, and not acceptable to the buyer if $S_b < 0$.

$$S_s \geq 0 \quad \rightarrow TR - TR_d \geq 0$$

$$\rightarrow (6) - (3) \geq 0$$

$$\rightarrow C_d \geq C_b - A_s ([1/(Q_b N)] - [1/(Q_d N_d)])$$

$$- ((N-1)Q_b - (N_d - 1) Q_d) H_s C_s /(2D) \qquad \text{--------} \quad (10)$$

$$S_b \geq 0 \quad \rightarrow TC - TC_d \geq 0$$

$$\rightarrow (2) - (5) \geq 0$$

$$\rightarrow C_d \leq (A_b D ((1/Q_d)-(1/Q_d)]$$

$$+ C_b [(Q_b H_b/2)+D]\}/(D+(H_b Q_d/2)) \qquad \text{--------} \quad (11)$$

For given Q_d, (10) imposes a bound on the discount price C_d, any value above which is acceptable to the seller; whereas (11) imposes a bound on C_d, any value below which is acceptable to the buyer. Intuitively, (10) tells us that the seller will not be prepared to reduce the unit selling price of his product below a certain value, and (11) that the buyer will not be prepared to pay more than a certain unit price for the product he purchases.

Let L denote the lower bound on C_d imposed by (10), and U upper bound imposed by (11). Then

$$L = C_b - A_s\{[1/(Q_b N)] - [1/(Q_d N_d)]\}$$
$$-\{(N-1)Q_b - (N_d - 1)Q_d\} H_s C_s/(2D) \qquad \text{--------- (12)}$$

$$U = \{A_b D[(1/Q_b) - (1/Q_d)] + C_b[(Q_b H_b/2)+D]\}/\{D+(H_b Q_d/2)\} \qquad \text{-----(13)}$$

A proposed discounting scheme is acceptable to both buyer and seller only if

$$L \leq C_d \leq U \qquad \text{---------(14)}$$

The following three propositions state some important characteristics of S_b, S_s, and $S_b + S_s$ which will be of use in the later sections of the paper.

<u>Proposition</u> <u>1</u>. For a given Q_d, let $L \leq U$. Then in the interval [L,U], S_b attains a maximum at $C_d = L$, and $S_s = 0$ at $C_d = L$. The former follows from the fact that in equation (8), which defines S_b, the coefficient of C_d is $-\{D+(Q_d H_b/2)\}$. Because the coefficient of C_d is negative, S_b increases as C_d decreases, and takes a maximum at the lowest value of C_d, $C_d = L$, in the interval [L, U]. The fact that $S_s = 0$ at L follows directly from the definition of L.

Proposition 2. For a given Q let $L_d \leq U$. Then in the interval [L,U], S_s attains a maximum at $C_d = U$, and at $C_d = U$, $S_b = 0$. The former follows from the fact that in equation (9), which defines S_s, the coefficient of C_d is D. Because the coefficient of C_d is positive S_s increases as C_d increases and takes a maximum at the highest value of C_d, $C_d = U$, in the interval [L,U]. The fact that $S_b = 0$ at at $C_d = U$ follows directly from the definition of U.

Proposition 3. For a given Q let $L_d \leq U$. Then in the interval [L,U], $S_s + S_b$ is a maximum at $C_d = L$, and minimum at $C_d = U$. To show this, observe that in (8) + (9) which defines $S_b + S_s$, the coefficient of C_d is $-D - (Q_d H_b/2) + D$, which is the same as $-Q_d H_b/2$. As this coefficient is negative, $S_b + S_s$ achieves a maximum for $C_d =$ L, and a minimum for $C_d = U$. Furthermore, in the interval [L,U] the function $S_s + S_b$ decreases linearly with respect to C_d, because the coefficient of C_d, as shown above, is negative and a constant. This variation is shown in Figure-1a.

3. Approximation used by Monahan and its conceptual implications

We emphasize here that the above model is exact for the assumptions made. It does not use the approximation of Monahan [3] and Lee and Rosenblatt [2]. We now show that the results of Monahan [3] and Lee and Rosenblatt [2] can be obtained as special cases of the above model. Furthermore, we also show that the approximation makes a serious difference in the conceptual implication of the model.

First, let us note that the approximation of Monahan [3] and Lee and Rosenblatt [2] consists in assuming that, for the buyer, the price discount does not change the inventory carrying cost per unit per year; that is, according to the approximation,

$$H_b C_b = H_b C_d \qquad \text{--------(15)}$$

To see the effect of the above approximation on the upperbound for C_d, let us re-write (11) as

$$C_d ((Q_d H_b/2) + D) \le A_b D ((1/Q_b) - (1/Q_d)) + C_b ((Q_b H_b/2) + D) \qquad \text{---(16)}$$

Expanding (16) and substituting $C_b H_b$ for $C_d H_b$, we get

$$C_d D \le A_b D[(1/Q_b) - (1/Q_d)] + C_b H_b [(Q_b/2) - (Q_d/2)] + C_b D \qquad \text{--- (17)}$$

Rearranging the terms in (17) we get

$$C_d \le C_b - A_b [(1/Q_b) - (1/Q_d)] - [C_b H_b/(2D)] (Q_d - Q_b) \qquad \text{------ (18)}$$

(18) gives us an approximate value of upperbound, $\hat{U}$ on C_d. To be consistent with Lee and Rosenblatt (1986) let us write Q_d as KQ_b, and use (18) to solve for $C_b - \hat{U}$. We get

$$C_b - \hat{U} = (2C_b H_b A_b/D)^{0.5} (K-1)^2/(2K) \qquad \text{------ (19)}$$

The above equation is the same as equation (2) for $d_k(K)$ of Lee and Rosenblatt [2], and equation (6) for $d_k(BE)$ of Monahan [3]. The expression for the lower bound L, of course, does not get affected by the approximation given by (15).

It is also important to note that under the approximation, the seller's increase in annual profit $\hat{S}_s$ remains the same as S_s ; but, the buyer's decrease in the annual cost S_b undergoes a change:

$$\hat{S}_b = A_b D[(1/Q_b) - (1/Q_d)] + D(C_b - C_d) - (H_b C_b/2)(Q_d - Q_b) \quad \text{----(20)}$$

Therefore, for a given Q_d and C_d, the sum of the benefits to the seller and buyer, according to the approximation is given by

$$\hat{S}_s + \hat{S}_b = S_s + \hat{S}_b = A_s D\{[1/(NQ_b)]-[1/(N_d Q_d)]\}$$

$$-(H_s C_s/2)\{Q_d(N_d-1) - Q_b(N-1)\} + A_b D\{(1/Q_b) - (1/Q_d)\}$$

$$-(H_b C_b/2)(Q_d - Q_b) \quad \text{------- (21)}$$

One important feature of the above equation is that it is independent of C_d. This implies that, according to the approximation, for a given Q_d, no matter which value C_d takes between U and L, the sum of the benefits to the buyer and seller is a constant. This feature is illustrated graphically in Figure-1b.

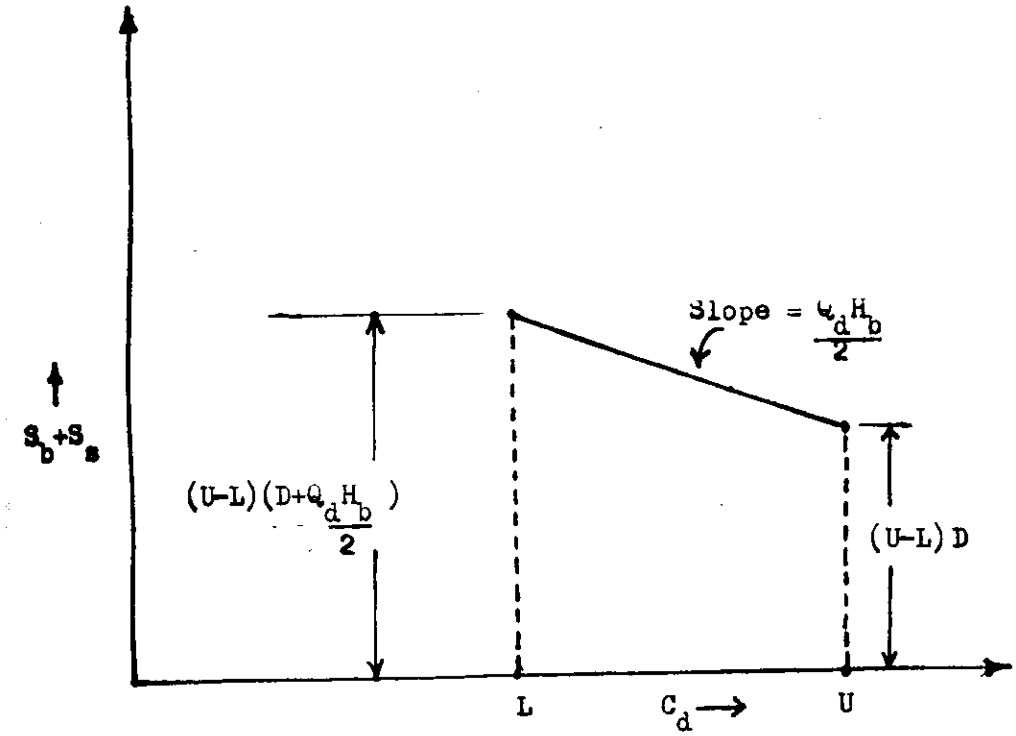


Figure 1a. Behaviour of $S_s + S_b$ for a given Q_d.

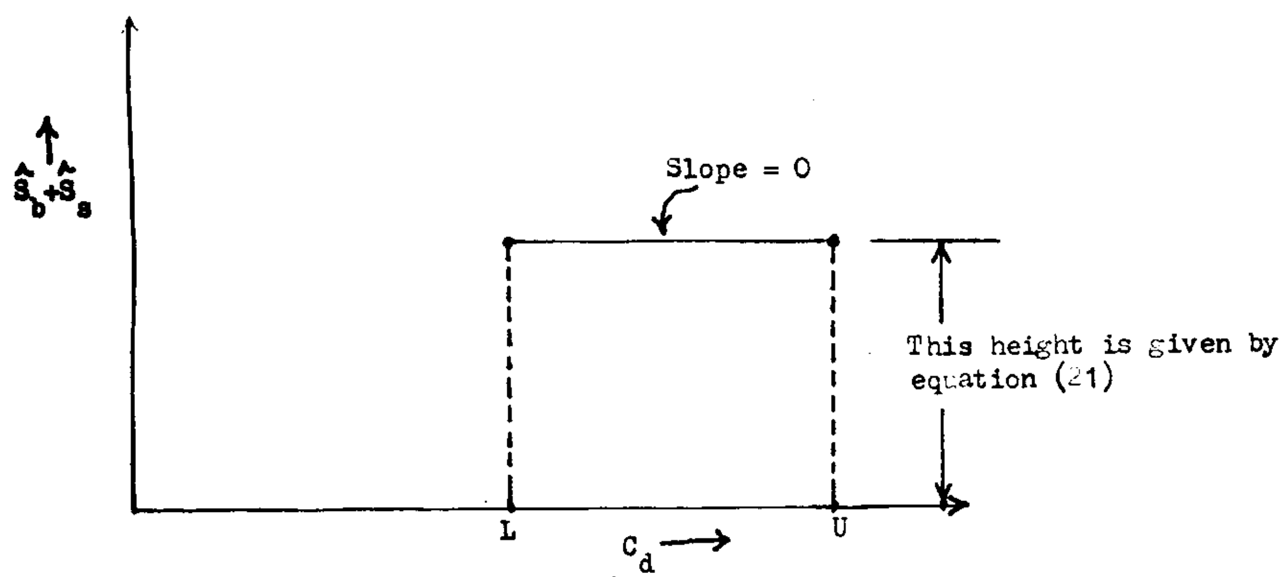


Figure 1b. Behaviour of $\hat{S}_s + \hat{S}_b$ for a given Q_d.

By comparing the above conclusion from the aproximate equations with the result stated in Proposition-3, we can easily see that the implications of the approximate analysis are at variance with those of exact analysis.

4. Optimal Discount

The problem of optimal price discount can be put in a clear perspective using the exact analysis. Earlier, we have shown that a given set of Q_d and C_d is acceptable to both buyer and seller if $L \leq C_d \leq U$. There can be several feasible pairs of Q_d and C_d each resulting in different gains for the buyer and seller. This makes it essential for us to analyse the impact of discounting from the seller's point of view as well as the buyer's point of view; additionally, it is also useful to carry out the analysis from the view-point of the total system. These three view-points are presented below.

Seller's view-point. If the seller is the sole decision maker in the discounting problem, then his objective will be to choose a feasible discounting scheme that maximizes his gain. In symbols, the seller's problem is to choose Q_d, C_d to

$$\text{Maximize} \quad TR_d - TR \qquad \text{-------} \quad (22)$$

$$\text{st} \quad L \leq C_d \leq U$$

As TR is constant, maximizing $TR - TR_d$ is equivalent to maximizing TR_d.

From proposition-1, we know that for every Q_d, $TR_d = TR$ at $C_d = L$. From (6), for fixed Q_d, the rate of variation of TR_d with respect to C_d is D. Thus for fixed Q_d, $TR_d = TR + (U-L)D$ at $C_d = U$ (See Figure -2a). As proposition-1 says that the maximum benefit to the seller, for a given Q_d, occurs at the corresponding U, we conclude that $TR_d = TR + (U-L) D$ is the maximum TR_d for a given Q_d. As TR and D are constants, among the different possible values of Q_d, that for which (U-L) is a maximum will maximize TR_d.

Therefore, to solve (22) we have to find Q^*_d, the Q_d for which (U-L) is a maximum; the optimal discounting scheme for the seller, therefore is to induce the buyer to increase his procurement lot size from Q_b to Q^*_d by offering him a reduced unit-price of U corresponding to Q^*_d.

<u>Buyer's</u> <u>Viewpoint</u>. The buyers objective will be to choose Q_d and C_d to

$$\text{Maximize} \quad TC - TC_d \qquad \text{------- (23)}$$

$$\text{st} \qquad L \leq C_d \leq U$$

As TC is a constant, maximizing $TC-TC_d$ is equivalent to minimizing TC_d.

From (5), for fixed Q_d, the rate of variation of TC_d with respect to C_d is $(H_b Q_d/2) + D$ (Figure 2b). We know from proposition-2 that, for a fixed Q_d, the minimum of TC_d occurs at $C_d = L$ and maximum at $C_d = U$. We know that the maximum of TC_d is the same as TC, the original cost. Therefore, the minimum of TC_d for a given Q_d is

$$TC - (U - L) \{D+(H_b Q_d/2)\} \qquad \text{--------} \quad (24)$$

Unlike in the case of the seller, the coefficient of (U-L) in the above minimum is a function of Q_d. Hence, there is no guarantee that the value of Q_d which maximizes (U - L) will minimize TC. Therefore, in general, it is not true that the buyers optimum and the sellers optimum occur at the same Q_d.

<u>System</u> <u>view-point</u>. If the objective of the discounting is to maximize the gains of both the buyer and the seller then the problem can be stated as one of choosing Q_d and C_d to

$$\text{Maximize } (TR_d - TR) + (TC - TC_d) \qquad \text{---------} \quad (25)$$

$$\text{st} \qquad L \leq C_d \leq U.$$

The above objective is equivalent to maximizing $TR_d - TC_d$. From (5) and (6), for a fixed Q_d, the rate at which $TR_d - TC_d$ varies with respect to C_d is $- H_b Q_d/2$ (Figure 2c). This result, combined with

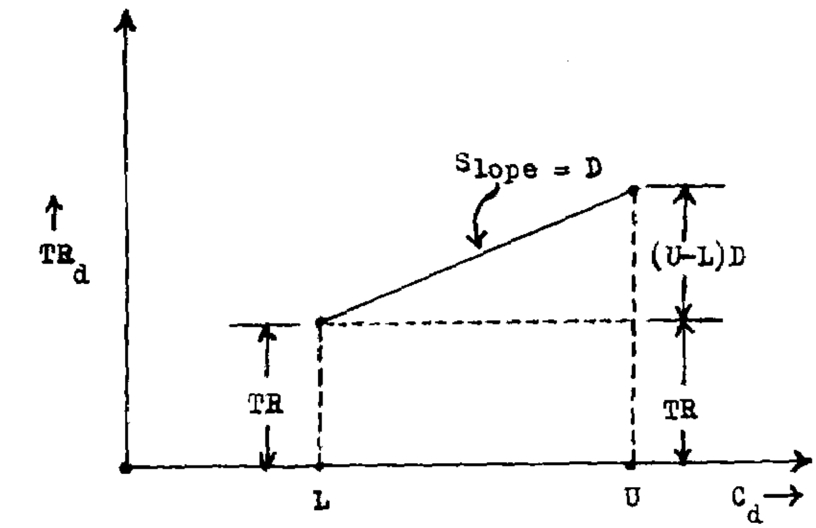


Figure-2a. Variation of TR_d for a given Q_d.

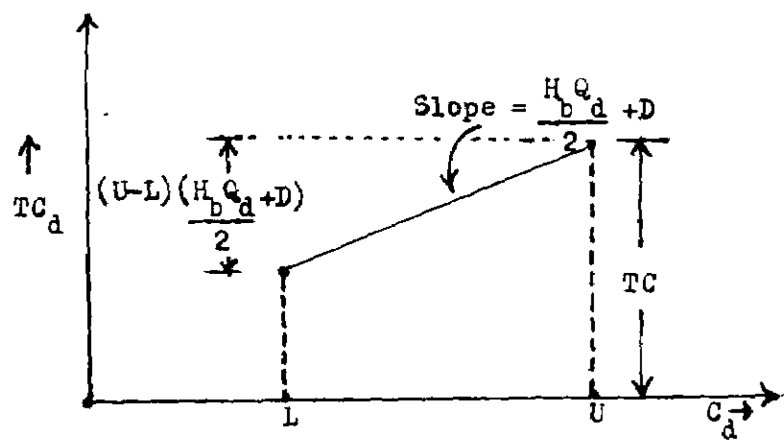
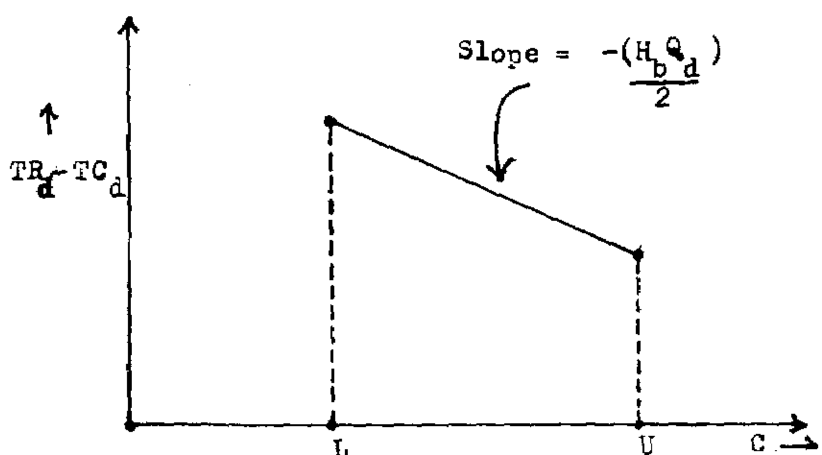


Figure-2b. Variation of TC_d for a given Q_d.



the results of proposition - 3, tells us that for a given Q_d, the maximum for the total system occurs at $C_d = L$. Thus, for a given Q_d, the optimum C_d is the same from both the buyer's view-point and the system's view-point. In fact, for a given Q_d, at $C_d = L$, both S_b and $S_b + S_s$ are equal to $(U-L)\{D + (H_b Q_d/2)\}$. Hence the global optimum for the buyer and the total system occur at the same Q_d and C_d.

Solution Procedures

Sellers Optimization

For a given Q_d, let $R(Q_d) = U-L$. As mentioned earlier, the objective of the seller is to find the Q_d for which $R(Q_d)$ is a maximum. From (12) we note that L is dependent on N_d which in turn is dependent on Q_d as given by (7). Therefore the seller's optimization involves choosing N_d and Q_d simultaneously. This can be done in two stages:

(1) For each N_d find $Q^*_d(N_d)$, which is the value of Q_d that maximizes $R(Q_d)$

(2) Compare $R(Q^*_d(N_d))$ for different values of N_d to choose the maximum value of $R(Q^*_d(N_d))$. The corresponding values of N_d, $Q^*_d(N_d)$, and U define the optimal solution for the seller. Let us denote those values by N^*_d (seller), Q^{**}_d (seller) and U* respectively.

It is not easy to find a closed form solution for $Q^*_d(N_d)$ by differentiation for a given N_d. Therefore in performing the first stage of computations stated above, we need to use a numerical search procedure. We give below several propositions that define the boundaries for Q_d in a given N_d, and those for N_d itself in the search method.

<u>Proposition 4</u>:

For any $N_d > 1$ the range of Q over which the search is needed is bound by $Q\,max(N_d)$ and $Q\,min(N_d)$ where

$$Q_{max}(N_d) = \{2\,A_s\,D/(N_d\,(N_d-1)\,H_s\,C_s)\}^{.5} \qquad \text{----------- (26)}$$

$$Q_{min}(N_d) = \{2\,A_s\,D/(N_d\,(N_d+1)\,H_s\,C_s)\}^{.5} \qquad \text{----------- (27)}$$

This is arrived at from (7), the expression for optimal N_d. It may also be noted that according to the above expressions

$$Q_{max}(N_d) = Q_{min}(N_d-1), \text{ for } N_d>1$$

Proposition 5: For $N_d = 1$

$$Q_{max}(1) = A_s [C_s - C_b + (A_s/NQ_b) + ((N-1) Q_b H_s C_s /2D)] \quad \text{-------(28)}$$

$$Q_{min}(1) = Q_{max}(2) \quad \text{-------(29)}$$

Expression (28) is obtained by imposing the constraint that $L \geq C_s$ and equating $N_d = 1$ in expression (12), whereas (29) follows from (26) & (27) directly.

Proposition 6: The search range of N_d is restricted to $N \leq N_d \leq 1$. It is obvious from the expression for N_d that as Q_d increases N_d decreases. Since the discount is offered only for a quantity higher than Q_b, N_d cannot be greater than N.

Buyer's optimization

The objective of the buyer is to find the Q_d for which (24) is minimum. This is equivalent to finding the Q_d for which (U-L) ((D+H_b Q_d/2)) is a maximum. A two stage search procedure similar to that of the seller's optimization can be used for this problem also; in the first stage, the above objective is maximized within each N_d by varying Q_d, whereas in the second, the different maxima found in first stage are compared to find the global maximum. However, the interval of search for N_d in this case is smaller than that of the seller's case, as shown below.

Proposition 7: The Q_d that maximizes buyer's savings cannot be less than the Q_d that maximises sellers savings. The proof is seen from expression (24) for the minimum total cost at a given Q_d.

For any $Q_d < Q^{**}_d$ (seller) the minimum total cost of the buyer is more than that at $Q = Q^{**}_d$ (seller). The corrollary to the above is that for finding out the buyer's optimum the search can be restricted to N^*_d (seller) $\leq N \leq 1$.

Propositions 4 to 7 thus define limits for the range of N and the range of Q within each N for the search.

System optimization: As noted earlier the optimum values for the system maximization are the same as that of buyer's optimization and hence no special search is needed.

5. Summary and Conclusion

In this paper we have presented the exact formulation for a discount pricing model. The exact formulation enabled us to analyze the problem from the view-points of the seller, the buyer, and the total system. We have shown that the optimal policies of the seller and the buyer need not necessarily be the same; and that, the optimal policies of the buyer and the system are the same. The actual discount that is offered in a real situation will be a result of negotiations between the buyer and the seller; and, in the present framework, the final discount depends on how far from his respective optimum each party is prepared to depart. The separate optima we discussed in section 4 serve as starting points in the negotiation process.

For situations where companies under the same group, or divisions under the same company transact among themselves, Lee and Rosenblatt [2] argue that pricing does not have any significance. This point of view is questionable because each entity in the above situation can be considered as a profit centre, and evaluated on its performance. In this context it helps to consider the buyer's view point also, along with the seller's, because buyer's view point coincides with the total system view point. The system optimization improves the overall performance without jeopradising the performance index of either the seller or the buyer. Therefore, it is important for the coordinating agency (parent company/corporate office) to motivate the selling and the buying companies to arrive at the discount in such a way that it optimizes the total gains of both the parties.

References

Hadley G and T.M. Whitin, (1963) Analysis of Inventory Systems, Prentice Hall, Inc. Englewolod Cliffs, N.J.

Lee H.L. and Rosenblatt M.J. (1986) A generalized quantity discount pricing model to increase suplier"s projects. Management Science 32(9), 1177-1185.

Monahan J.P. (1984) A quantity discount pricing model to increase under projects. Management Science 30(6), 720-726.

****