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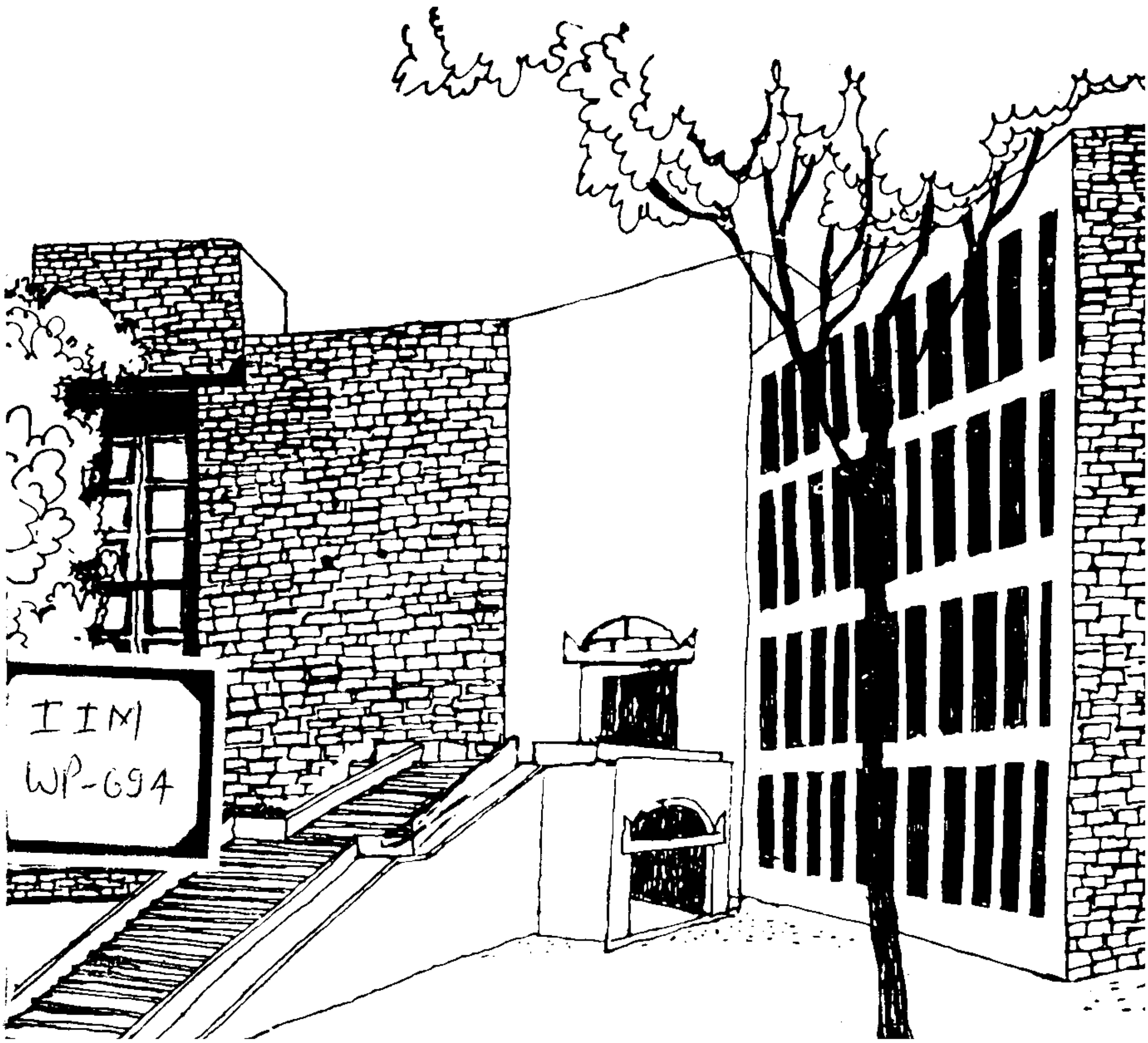

THE SIMPLEST PROBLEM IN OPTIMAL CONTROL REVISITED

BY

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ABSTRACT

In this paper, we define and study the simplest problem in optimal control theory, with differential inequalities determining the evolution of state variables. We prove necessary and sufficient conditions for optimality.

1. Introduction :

The simplest problems in most areas, are often regarded as overworked, and therefore necessarily barren. Thus it might come as a surprise to note that we intend to break fresh ground with the simplest problem of optimal control theory - or else why revisit it?

In optimal control problems, variables are divided into two classes, State variables and Control variables. The movement of State variables is governed by first order differential equations. The simplest control problem is one of selecting a piecewise continuous Control function $u(t)$, $t_0 \leq t \leq t_1$, to

$$\max \int_{t_0}^{t_1} f(t, X(t), u(t))\, dt \qquad (1)$$

$$\text{Subject to } x'(t) = g(t, X(t), u(t)) \qquad (2)$$

$$t_0,\ t_1,\ x(t_0) = x_0 \text{ fixed; } x(t_1) \text{ free} \qquad (3)$$

Here f and g are assumed to be known and continuously differentiable functions of three independent arguments, none of which is a derivative.

The Control variable u (t) must be a piecewise continuous function of time. The State variable x (t) changes over time according to the differential equation (2) governing its movement. The Control u influences the objective (1), both directly (through its own value) and indirectly through its impact on the evolution of the State variable x (that enters the objective (1)). The highest derivative appearing in the problem formulation is the first derivative, and it appears only as the left side of the State equation (2). (Equation (2) is sometimes also called the transition equation). A problem involving higher derivatives can be transformed into one in which the highest derivative is the first.

In this paper, we propose to study optimal control problems, which are similar to the one already defined above, except that the evolution of the state variables are determined by differential inequalities instead of differential equations. In this framework, we obtain necessary conditions which a trajectory of state variables and a trajectory of control variables must necessarily satisfy, in order to qualify as an optimal solution to the revised problem.

2. Differential Inequalities :

A function $x(t)$ is said to be a solution of the differential inequality

$$x' > h(t, x) \text{ or } x' \geq h(t, x)$$

on an interval I if it is differentiable and satisfies

$$x'(t) > h(t, x(t)) \text{ or } x'(t) \geq h(t, x(t)),$$

respectively, for every t in I. For example, the function $x(t) = \tan t$ is a solution of the differential inequality $x' > x^2$ on the interval $(-\frac{\pi}{2}, \frac{\pi}{2})$ since $x'(t) = 1 + \tan^2 t$.

In view of the applications of the inequalities, it would be useful if the foregoing definitions are extended. For any scalar function $x(t)$, the upper- and lower-right derivative, $D^+ x$ and $D_+ x$ are defined by

$$D^+ x(t) = \limsup_{h \to 0^+} \frac{(x(t+h) - x(t))}{h},$$

$$D_+ x(t) = \liminf_{h \to 0^+} \frac{(x(t+h) - x(t))}{h}$$

Similarly, the upper-and lower-left derivative of $x(t)$, $D^- x$ and $D_- x$, are given by

$$D^- x(t) = \limsup_{h \to 0^-} (x(t+h) - x(t))$$

$$D_- x(t) = \liminf_{h \to 0^-} \frac{(x(t) + h) - x(t))}{h}$$

These derivatives are usually referred to as Dini's derivatives.

A function $X(t)$ is said to be a solution of, let us say, the differential inequality $D_+ X > h(t, X)$ on an interval I if $D_+ X(t) > h(t, X(t))$ for every t in I. When $D_+ X(t) = D^+ X(t)$, the right-hand derivative of $X(t)$ exists and is often denoted $X'_+(t)$. Likewise, $X'_-(t)$ represents the left-hand derivative of $X(t)$.

Consider the initial value problem

$x' = h(t, X),\ X(t_0) = X_0,$

where $h \in C[\Omega, \mathbb{R}]$, Ω being an open set in $\mathbb{R}^2$. Let $J_1 = (t_0, t_0 + a)$,

a 0.

<u>Definition 1</u> : A function $y \in C(J_1, \mathbb{R})$ is said to be an upper (or a lower) function with respect to the initial value problem if $y'_+(t)$ exists and satisfies the differential inequality $y'_+(t) > h(t, y(t))$ $[$ or $y'_+(t) \leq h(t, y(t))]$ on J_1.

If $Y'_+(t) = y'_-(t)$, we say that $y \in C(J_1, \mathbb{R})$ is a differentiable upper (or a lower) function.

3. <u>Optimal Control Problems With Differential Inequalities</u> :

The prototype optimal control problem with differential inequalities, is of the form

$$\max \int_{t_0}^{t_1} f(t, X(t), u(t))\, dt \qquad (4)$$

$$\text{Subject to } X'(t) \leq g(t, X(t), u(t)) \qquad (5)$$

$$t_0,\ t_1,\ X(t_0) = x(t_1) \text{ free} \qquad (6)$$

To find necessary conditions that a maximizing solution $u^*(t)$, $X^*(t)$, $t_0 \leq t \leq t_1$, to problem (4) - (6) must obey, we follow a procedure reminiscent of solving a nonlinear programming problem with Lagrange multipliers.

First let us suppose that for some $\underline{t} \in [t_0, t_1]$,

$$X^{*'}(\underline{t}) < g(\underline{t}, X^*(\underline{t}), u^*(\underline{t}))$$

Then by continuity of the solutions, there exists an interval J, open in the relative topology of $[t_0, t_1]$, containing $\underline{t}$, such that

$$X^{*'}(t) < g(t, X^*(t), u^*(t)) \text{ for all } t \in J.$$

Let $J_1, \ldots, J_n$ be the non-overlapping intervals, possibly containing t_0 or t_1 or both, where

$$X^{*'}(t) < g(t, X^*(t), u^*(t))$$

For $t \in (J_1 \cup \ldots \cup J_n)^c$,

$$X^{*'}(t) = g(t, X^*(t), u^*(t))$$

For any function X, u satisfying (5) and (6))

$$\int_{t_0}^{t_1} f(t, X(t), u(t))\, dt = \sum_{i=1}^{n} \int_{J_i} (f(t, X(t), u(t))\, dt + \int_{(J_1 \cup \ldots \cup J_n)^c} f(t, X(t), u(t))$$

Let us consider each J_i in turn. Let $J_i = (\tau_i, \tau_{i+1})$. We consider a one parameter family of comparison controls on J_i, given by $u^*(t) + ah(t)$, where $u^*(t)$ is the optimal control, h (t) in some fixed continuously differentiable function, and a is a parameter. For any $J \subsetneq \operatorname{int}(J_i)$ there exists, $\alpha > 0$ such that for all $a, -\alpha \le a < \alpha$

$$X^{*'}(t) + a\, l'(t) < g(t, X(t) + a\, l(t), u(t) + a\, h(t))$$

where l is some fixed continuously differentiable function.

Once again we are exploiting the continuity property of g.

$\therefore \int_J (f(t, X^*(t) + a\, l(t), u^* + a\, h(t))\, dt$, as a function of 'a',

attains its maximum at a = 0

Differentiating with respect to 'a' and evaluating at a = o, we obtain,

$$\int_J \left[f_x (t, X^* (t), u^* (t)) \ell (t) + f_u (t, X (t), u (t)) h (t) \right] dt = o$$

This being true for every possible l (t) and h (t), we obtain,

$$f_x(t, X^* (t), u^*(t)) = o, \; f_u (t, X^*(t), u^* (t)) = o \text{ on } J$$

However, $J \subseteq$ int. (J_i) is arbitrary

$\therefore f_x (t, X^* (t), u^*(t)) = o, \; f_u (t, X^*(t), u^* (t)) = o$ for all $t \in J_i$.

$i = 1,\ldots,n$.

Note, both h and l are required to satisfy $h (I_i) = h (I_{i+1}) = l (I_i) = l(I_{i+1}) = 0$

That leaves us with, $(t_o, t_1) \setminus (J_1 \cup \ldots \cup J_n)$. If the set just defined has empty interior, then it is a subset of $\partial J_1 \cup \ldots \cup \partial J_n$ where if A is a set, ∂A denotes its topological boundary, and hence the endpoints of the sets J_i, in this case. In such a situation, $f_x (t, X^*(t), u^*(t)) = o$, $f_u (t, X^*(t), u^*(t)) = o$ for all $t \in (t_o, t_1)$, by continuity of the first partial derivatives of f.

If however $(t_o, t_1) \setminus (J_1 \cup \ldots \cup J_n)$ has non empty interior, it must contain an open interval I, On I

$$X^{*\prime} (t) = g (t, X^* (t), u^*(t))$$

<u>Case 1</u> : Let us suppose $t_1 \notin$ I. In this case it follows from standard theorems in optimal control theory, that there is a continuously differentiable function λ (t) such that u^*, X^*, λ simultaneously satisfy the <u>state equation</u>

$$X' (t) = g (t, X (t), u (t))$$

the <u>multiplier equation</u>

$$\lambda' (t) = - \left[f_x (t, X (t), u (t)) + \lambda(t) g_x (t, X (t), u (t)) \right]$$

and the <u>optimality condition</u>

$$f_u (t, X (t), u (t)) + \lambda(t) g_u (t, X (t), u (t)) = 0$$

for $t \in$ I.

Case 2 : $t_1 \in I$. In this case, once again all the conditions in Case 1 are satisfied; in addition the transversality condition $\lambda(t_1) = 0$ holds. This again follows from standard theorems in optimal control theory.

We may thus formulate the following theorem :

Theorem 1: Let $u^*(t)$, $x^*(t)$, $t_0 \leq t \leq t_1$ solve problem (4) - (6)

Then if at $t \in [t_0, t_1]$,

$$x^{*\prime}(t) < g(t, x(t), u(t))$$

then

$$f_x(t, x^*(t), u^*(t)) = 0, \; f_u(t, x^*(t), u^*(t)) = 0$$

The same is true if $t \in \text{cl}\{t' \in [t_0, t_1] / x^{*\prime}(t') < g(t', x^*(t'), u^*(t'))\}$

If $t \in \text{int.}\left((t_0, t_1) \setminus \{t' \in [t_0, t_1] / x^{*\prime}(t') < g(t', x^*(t'), u^*(t'))\}\right)$, then there exists a continuously differentiable function $\lambda(t)$ such that u^*, x^*, λ simultaneously satisfy

$$\frac{\partial H}{\partial u} = f_u + \lambda g_u = 0$$

$$\lambda'(t) = -\partial H/\partial x = -(f_x + \lambda g_x)$$

$$x = \frac{\partial H}{\partial \lambda} = g,$$

Where,

$$H(t, x(t), u(t), \lambda(t)) = f(t, x, u) + \lambda g(t, x, u)$$

Further if t_1 belongs to the set then $\lambda(t_1) = 0$

If $x^{*\prime}(t) = g(t, x(t), u(t))$ we say that the solution is tight at time 't'.

What the above theorem states is everywhere that the solution is not tight, th partial derivatives of the inst-antaneous objective function vanishes. Everywh else, the familiar conditions of optimal control theory are satisfied.

For a maximization problem it is also necessary that every point 't' where the solution is not tight $u^*(t)$ maximize $f(t, x^*(t), u)$ with respect to u. Thus $f_{uu}(t, X^*(t), u^*(t)) \leq 0$ is necessary for maximization if the solution is not tight at time 't'. It is also necessary that $f(t, X, u^*(t))$ be maximized by $X^*(t)$, if the solution is not tight at time 't'. So we require $f_{XX}(t, X^*(t), u^*(t)) \leq 0$.

At a point 't', where the solution is tight, it is necessary that $u^*(t)$ maximize $H(t, X^*(t), u, \lambda(t))$ with respect to u. Thus $H_{uu}(t, X^*, u^*, \lambda) \leq 0$ is necessary for maximization, if the solution is tight at 't'.

Sufficiency:-

When are the necessary conditions for optimality both necessary and sufficient? In nonlinear programming, the Kuhn-Tucker necessary conditions are also sufficient provided that concave objective function is to be maximized over a closed convex region. Analogous results obtain for optimal control problems with differential inequalities.

Theorem 2 : Suppose that $f(t, X, u)$ and $g(t, X, u)$ are both differentiable concave functions of X and U in the problem

$$\max \int_{t_0}^{t_1} f(t, X, u)\, dt. \qquad (4)$$

$$\text{subject to } X' \leq g(t, X, u) \qquad (5)$$

$$t_0,\ t_1,\ X(t_0) = X,\ \text{fixed};\ X(t_1)\ \text{free} \qquad (6)$$

The argument t of $x(t)$ and $u(t)$ will frequently be suppressed. Suppose that the functions X^*, u^*, λ satisfy the necessary conditions

$$f_u(t, X, u) + \lambda g_u(t, x, u) = 0$$

$$\lambda' = - f_x(t, X, u) - g_x(t, X, u),$$

at a point 't' where the solutions is tight, with

$\lambda(t_1) = 0$, if the solution is tight at t_1

and $f_u(t, X, u) = 0 = f_x(t, X, u)$

at a point 't' where the solution is not ~~gix~~ tight. Suppose further that X and λ are continuous with

$$\lambda(t) \geq 0$$

for all points 't' at which the solution is tight in case $g(t, X, u)$ is non-linear in X or in u, or both. Then the functions X^*, u^* solve the problem given by (4), (5) and (6) among all functions X, u which have the same set of tight time points, and for which $X(t) = X^*(t)$ if t belongs the boundary of the set of tight time points. For $t = t_1$, the latter restriction is not required.

Proof :

Let J be the set of time points at which the solution is not tight and $J^c = [t_0, t_1] \setminus J$, the set of time points at which the solution is tight. Let X, u be functions which have the same set of time points as X^*, u^* where the tightness condition is satisfied. Let f^*, g^* and so on denote functions evaluated along the feasible path (t, X^*, u^*). Then we must show that

$$D = \int_{t_0}^{t_1} (f^* - f)\,dt \geq 0$$

$$\text{Let } D_J = \int_J (f^* - f)\,dt.$$

$$D_{J^c} = \int_{J^c} (f^* - f)\,dt.$$

The concavity of f and the fact that $f_x^* = 0 = f_u^*$ implies that $\forall\, t \in J$,

$$f(t, X^*(t), u^*(t)) \geq f(t, X(t), u(t))$$

$$\therefore D_J \geq 0.$$

Since f is a concave function of (x, u) we have

$$f^{*} - f \geq (x^{*} - x)\, f_x^{*} + (u^{*} - u)\, f_u^{*}, \text{ and therefore}$$

$$D_J c \geq \int_{J^c} \left[(x^{*} - x)\, f_x^{*} + (u^{*} - u)\, f_u^{*} \right] dt$$

$$= \int_{J^c} \left[(x^{*} - x)\, (-\lambda g_x^{*} - \lambda') + (u^{*} - u)\, (-\lambda g_u) \right] dt$$

$$= \int_{J^c} \lambda \left\{ g^{*} - g - (x^{*} - x)\, g_x^{*} - (u^{*} - u)\, g_u^{*} \right\} dt \geq 0$$

(on integrating terms involving λ' by parts and using the fact that $x(t) = x^{*}(t)$ if t is on the boundary of the set of time points which are tight).

Note if the function g is linear in X, u, then λ may assume any sign at a point where the solution is tight. Otherwise concavity of g in x and u and nonnegativeness of λ guarantees the last inequality.

5. Conclusion and Envoy for Further Research :

A good source of information about differential inequalities is the book by Lakshmikantham and Leela [2]. Background material on necessary conditions for the simplest problem in optimal control can be obtained from Varaiya [5]. The prototype sufficiency theorem for optimal control with differential equations appears in Managsarian [3].

Further sufficiency theorems for optimal control problems have been proved by Arrow and Kurz [1] and Seierstad and Sydsaeter [4]. Whereas, the Arrow type sufficiency theorem can be easily adadpted to our framework, with the provise that $(x, u) \mapsto f(t, x, u)$ be a concave function in X and u at every time point 't' where candidate x^{*}, u^{*} is not tight, it is an open question whether the more general sufficiency theorems of Seierstad and Sydsaeter can be accommodated in our framework.

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