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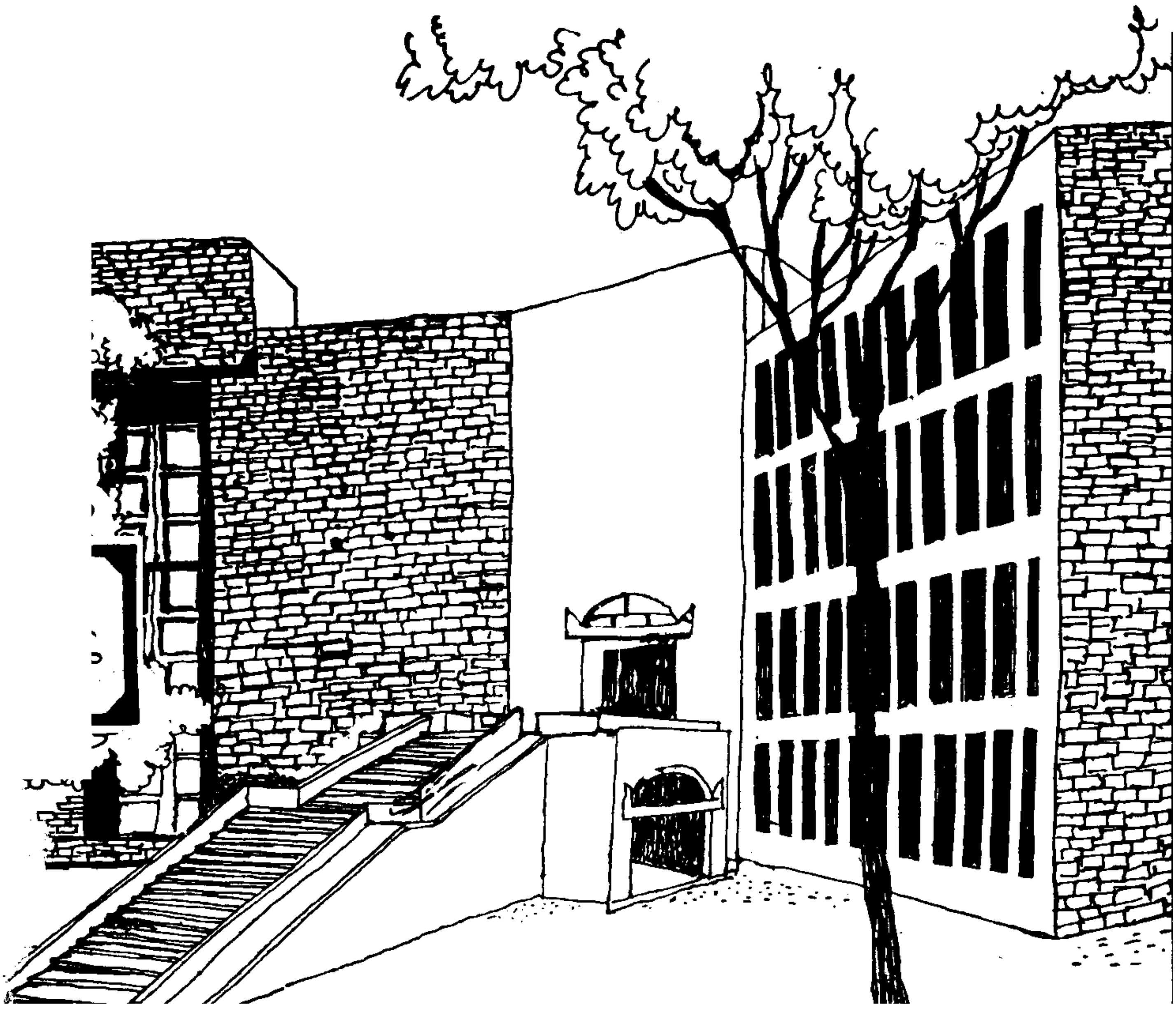

OPTIMAL CONSUMPTION PLANS WITH UNCERTAIN PLANNING PERIODS

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A B S T R A C T

In this paper we study the one sector optimal growth model with uncertain planning horizons. We prove the non-existence of steady states, and the dependence of optimal capital stock at time 't' on the conditional probability of a 't' period planning horizon given that the planning process does not terminate before time 't'. We illustrate our results using a consumption optimal growth model and Cobb-Douglas technology.

A centrally planned economy seeks the consumption rate for each moment of time that will maximize its discounted utility stream over a planning period of length T. The utility of consumption U (C(t)) at each moment t is a Known increasing concave function (nonincreasing marginal utility of consumption): $U' > 0$ and $U'' \leq 0$. Future utility is discounted at rate r. The objective is

$$\max \int_0^T e^{-rt} U(C(t))\,dt \qquad (1)$$

subject to a technology constraint. From a stock of capital equal to K, an output can be produced at rate F(K). The production function F is assumed to be twice continuously differentiable, increasing, and concave. This output can be consumed, yielding immediate satisfaction, or it can be invested to augment the capital stock and hence future productive capacity. Output F(K) is therefore the sum of consumption C and investment K' perfectly durable, but decays at a constant proportionate rate b, then reinvestment at rate bK(t) is required to keep its stock intact and therefore,

$$F(K) = C + K' + bK \qquad (2)$$

This is the familiar framework of optimal growth theory which has been developed by Arrow and Kurz(1970) amongst many others, too numerous to mention.

Our objective is to obtain the optimal path of capital accumulation when the planning period is uncertain.

Let $G(t)$ be the (known) probability of the planning process extending upto time 't'. Then $G'(t)$ is the probability density of a 't' period planning horizon. G is a nondecreasing function with $G(o)=o$ and $G(t_1) = 1$; the planning horizon cannot extend beyond t_1 years definitely. The social planners problem now reduces to

$$\max \int_0^{t_1} \left[\int_0^T e^{-rt}\, U\left[F(K(t)) - K'(t) - bK(t)\right] dt \right] G'(T)\,dT. \quad (3)$$

Integrating by parts we obtain that,

$$\int_0^{t_1} \left[\int_0^T e^{-rt} U\left[F(K(t)) - K'(t) - bK(t)\right] dt \right] G'(T)\, dT$$

$$= \left\{ \int_0^T e^{-rt} U\left[F(K(t)) - K'(t) - bK(t)\right] dt \right\} G(T) \Bigg|_0^{t_1}$$

$$- \int_0^{t_1} e^{-rT} U\left[F(K(T)) - K'(T) - bK(T)\right] G(T)\, dT$$

$$= G(t_1)\int_0^{t_1} e^{-rt}U\left[F(K(t)) - K'(t) - bK(t)\right] dt$$

$$- \int_0^{T_1} e^{-rt}\, U\left[F(K(t)) - K'(t) - bK(t)\right] G(t)dt$$

$$= \int_0^{t_1} e^{-rt}U\left[F(K(t)) - K'(t) - bK(t)\right]\left[1-G(t)\right] dt$$

Suppose for simplicity,

$$U(C(t)) = C(t).$$

Then, our planning problem reduces to

$$\max \int_0^{t_1} e^{-rt}\left[F(K(t)) - K'(t) - bK(t)\right]\left[1-G(t)\right] dt$$

$$\text{s.t.}\quad K(0) = K_0.$$

The Euler Equation for this problem is:

$$e^{-rt}\left[F'(K) - b\right]\left[1-G\right] = \frac{d}{dt}\left[-e^{-rt}\left\{1-G(t)\right\}\right]$$

$$= - e^{-rt}\left[-G'\right] + re^{-rt}\left\{1-G\right\}.$$

$$\text{or} \quad \left[F'(K(t))-b\right]\left[1-G(t)\right] = r\left[1-G(t)\right] + G'(t)$$

$$\text{or} \quad \left[1-G(t)\right]\left\{F'(K(t)) - b - r\right\} = G'(t)$$

$$\text{or} \quad F'(K(t)) - b - r = \frac{G'(t)}{1-G(t)}$$

The transversality condition requires $1-G(t_1) = 0$, which is automatically satisfied, and therefore provides no new information.

If the technology is of the form $F(K(t)) = \left[K(t)\right]^{\alpha}$, $0 < \alpha < 1$,

then,

we obtain,

$$\alpha\left[K(t)\right]^{\alpha-1} - b - r = \frac{G'(t)}{1-G(t)}$$

$$\text{or} \quad \left[K(t)\right]^{\alpha-1} = \frac{1}{\alpha}\left\{b + r + \frac{G'(t)}{1-G(t)}\right\}$$

$$\text{or} \quad K(t) = \left[\frac{1}{\alpha}\left\{b + r + \frac{G'(t)}{1-G(t)}\right\}\right]^{\frac{1}{\alpha-1}}$$

This is satisfied by our program of capital accumulation provided,

$$K_0 = \left[\frac{1}{\alpha}\left\{b + r + G'(0)\right\}\right]^{\frac{1}{\alpha-1}}$$

Let K^* be a steady state capital stock. Then

$$F'(K^*) = b + r + \frac{G'(t)}{1-G(t)} \quad \forall\, t \geq 0$$

i.e. $F'(K^*) = b + r - \frac{d}{dt} \log \left[1-G(t)\right] \quad \forall\ t_1 \geq t \geq 0$.

or $\frac{d}{dt} \log \left[1-G(t)\right] = b + r - F'(K^*)$.

$\therefore \log \left[1-G(t)\right] = \left[b + r - F'(K^*)\right] t + \gamma$

where γ is a constant.

Now, $G(o) = o \Rightarrow$

$0 = \log 1 = \gamma$.

$\therefore \log \left[1-G(t)\right] = \left[b + r - F'(K^*)\right] t$

Now $G(t_1) = 1 \Rightarrow \lim_{t \to t_1} \log \left[1-G(t)\right] = -\infty$.

But $\lim_{t \to t_1} \left[b + r - F'(K^*)\right] t = b + r - F'(K^*)\, t_1$ which is finite if K^* exists.

This proves the nonexistence of an optimal steady state capital stock.

Since convergent paths of capital accumulation, tend towards a steady state, we also obtain the nonexistence of a convergent path of capital accumulation from the above analysis.

Our main conclusion is that along the optimal path of capital accumulation the capital stock at any time 't' depends on the conditional probability of a 't' period planning horizon given that the planning period does not terminate before time 't'.

REFERENCE:

1. ARROW, K.J. and M. KURZ(1970): "Public Investment, The Rate of Return and Optimal Fiscal Policy", Baltimore: Johns Hopkins, 1970, Chapter 2.