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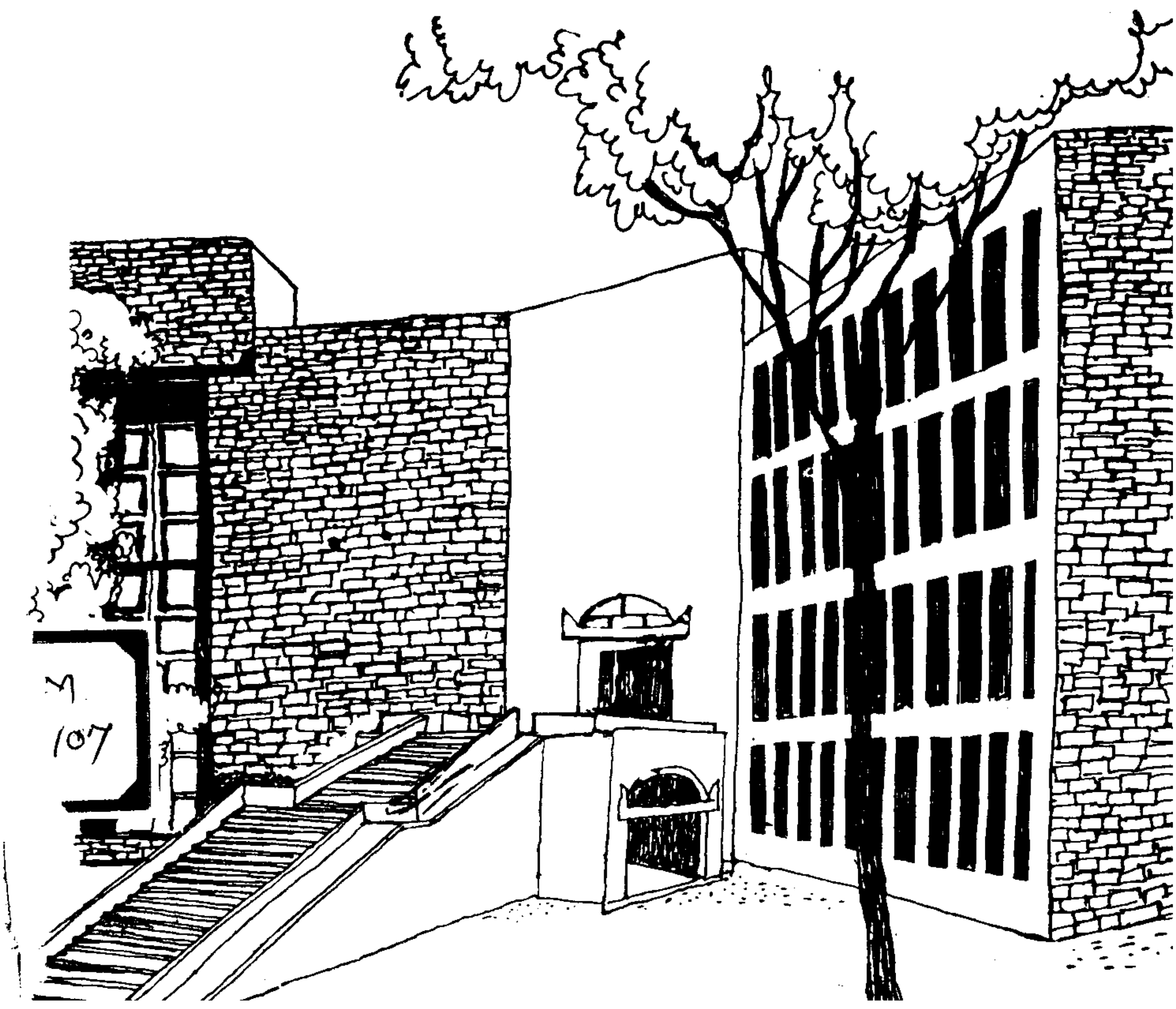

A NOTE ON THE SECOND ORDER CONDITIONS FOR ISOPERIMETRIC PROBLEMS IN DYNAMIC OPTIMIZATION

By

Somdeb Lahiri

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INDIAN INSTITUTE OF MANAGEMENT
AHMEDABAD-380015
INDIA

A B S T R A C T

In this paper we obtain a clear statement of the second order necessary condition for isoperimetric problems in dynamic optimization.

A dynamic optimization problem may be subject to an integral constraint:

$$\max \int_{t_o}^{t_1} F(t, x, x')dt \tag{1}$$

$$\text{subject to} \int_{t_o}^{t_1} G(t, x, x')dt = B, \quad x(t_o) = x_o, \quad x(t_1) = x_1 \tag{2}$$

where F and G are twice continuously differentiable functions and B is a given number.

Example:

$$\max \int_{o}^{T} e^{-rt} P_t(x)dt \tag{3}$$

$$\text{subject to} \int_{o}^{T} x\, dt = B \tag{4}$$

where $x(t)$ is the rate of extraction of a resource, B the initial endowment of the resource, and $P_t(x)$ the profit rate at t if the resource is extracted and sold at rate $x(t)$.

Typically the first-order necessary conditions for the above category of problems are obtained by viewing them as extensions of static equality constrained optimization problems involving the Lagrangian multiplier technique.

A workable method outlined in Kamien and Schwartz (1981; pg. 44-45) proceeds as follows:

Appending constraint (2) to (1) by an undetermined multiplier λ, we obtain for any admissible function x (i.e. any function x satisfying (2))

$$\int_{t_o}^{t_1} F(t, x, x')dt = \int_{t_o}^{t_1} [F(t, x, x') - \lambda G(t, x, x')] dt + \lambda B \tag{5}$$

The integral on the left attains its extreme values with respect to x just where the integral on the right does; λ then is chosen so that (2) is satisfied. The Euler equation for the integral on the right is

$$F_x - \lambda G_x = \frac{d}{dt}(F_{x'} - \lambda G_{x'}) \tag{6}$$

The existing literature however lacks a clear statement on the second order necessary condition for a maximum. The following lemma asserts the required criterion.

Lemma: Let λ^* be the multiplier so that (2) is satisfied in the argument leading upto (6). If x^* is an optimal solution, then

$$F_{x'x'}(t, x^*(t), x^{*\prime}(t) - \lambda^* G_{x'x'}(t, x^*(t), x^{*\prime}(t)) \leq 0$$

Proof: Let λ be arbitrary and let $x^*_\lambda(.)$ be the extremizer of

$$\int_{t_o}^{t_1} [F(t, x, x') - \lambda G(t, x, x')]dt$$

Let λ^* be such that

$$\int_{t_o}^{t_1} G(t, x^*_{\lambda^*}(t), x^{*\prime}_{\lambda^*}(t))dt = B$$

$$\therefore \int_{t_0}^{t_1} [F(t, x^*_{\lambda^*}, x^{*\prime}_{\lambda^*}) - \lambda^* G(t, x^*_{\lambda^*}, x^{*\prime}_{\lambda^*})]dt \geq$$

$$\int_{t_0}^{t_1} [F(t, x, x') - \lambda^* G(t, x, x')]dt$$

for any other admissible x(i.e. $\int_{t_0}^{t_1} G(t, x, x')dt = B$)

$$\therefore \int_{t_0}^{t_1} F(t, x^*_{\lambda^*}, x^{*\prime}_{\lambda^*})dt \geq \int_{t_0}^{t_1} F(t, x, x')dt, \text{ whenever}$$

$$\int_{t_0}^{t_1} G(t, x, x')dt = B.$$

Our notation suggests that $x^*_{\lambda^*} \equiv x^*$ and

$$\int_{t_0}^{t_1} [F(t, x^*, x^{*\prime}) - \lambda^* G(t, x^*, x^{*\prime})]dt \geq$$

$$\int_{t_0}^{t_1} [F(t, x, x') - \lambda^* G(t, x, x')]dt$$

whenever $x(t_0) = x_0$, $x(t_1) = x_1$.

Hence the second order necessary condition which is equivalent to the Legendre condition is applicable at x^* for the problem

$$\max \int_{t_0}^{t_1} [F(t, x, x') - \lambda^* G(t, x, x')]dt$$

subject to $x(t_0) = x_0$, $x(t_1) = x_1$.

Therefore by Kamien and Schwartz [(1981), pg. 39],

$$F_{x'x'}(t,\ x^*(t),\ x^{*\prime}(t)) - \lambda^*\ G_{x'x'}(t,\ x^*(t),\ x^{*\prime}(t)) \leq 0$$

as was required to be proved.

REFERENCE:

1. KAMIEN, M.I. and N. Schwartz: [1981]: 'Dynamic Optimization: The Calculus of Variations and Optimal Control in Economics and Management', North Holland, New York, Oxford.