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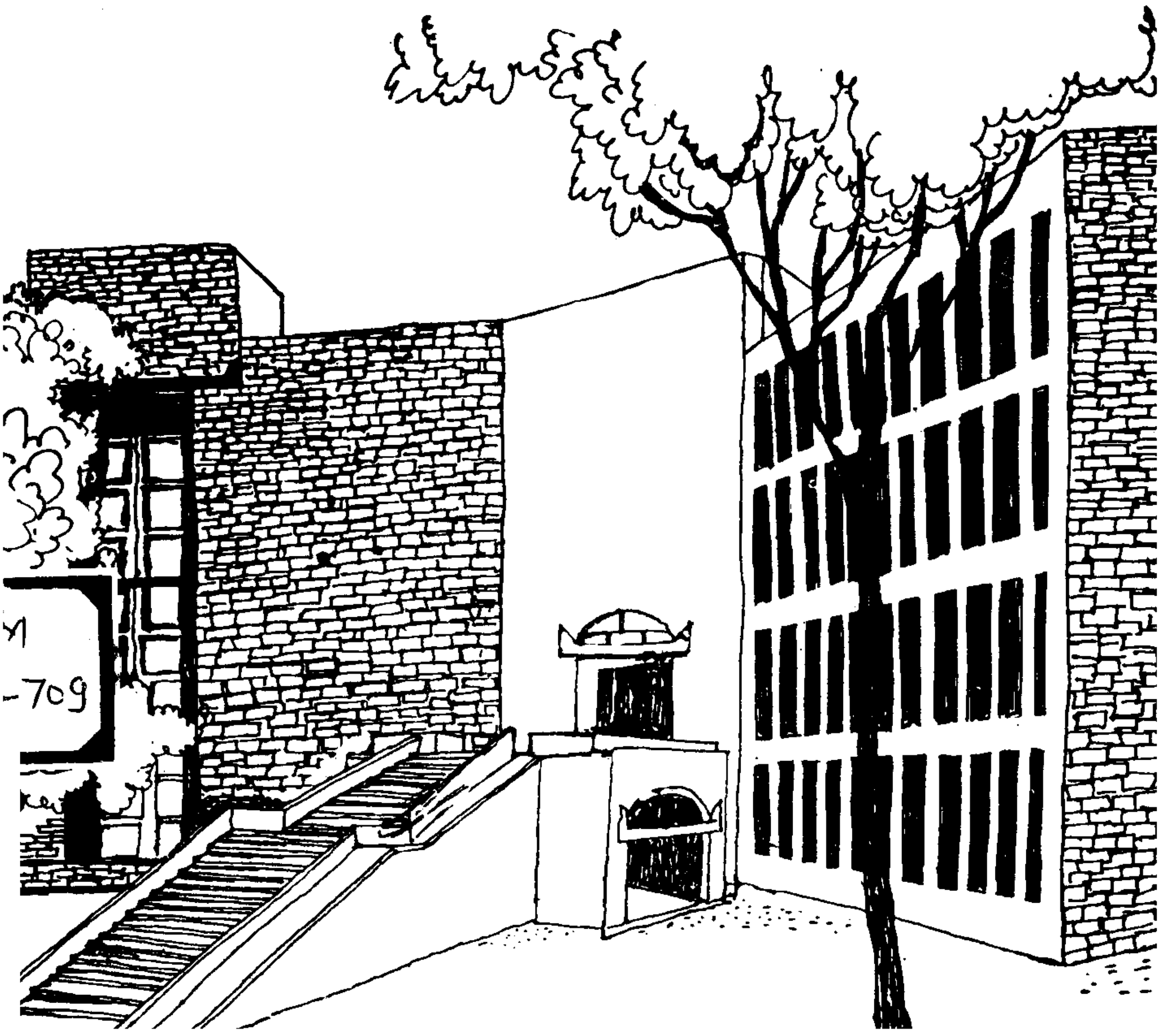

A GENERALIZATION OF THE NASH BARGAINING SOLUTION IN TWO PERSON CO-OPERATIVE GAMES

By

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A B S T R A C T

Existence and characterization of a generalized bargaining solution incorporating preferential treatment, is discussed. Similar results pertaining to variable reference point games are motivated.

1. **Introduction**:- The (fixed threat) bargaining model of Nash (1950) deals with two person games in which the players each obtain fixed utility levels if they fail to make an agreement. There is a feasible set of outcomes that they can achieve if they make an agreement; however, in the absence of agreement, there is nothing a player can do to help or hurt either himself or the other player. The rules defining Nash's solution concept are considered fair and leads to a desirable agreement.

However most bargaining situations in society are partial towards one or other player, according to the tenets enshrined in the social code of laws. It is to incorporate this possible asymmetry in outcomes, resulting from preferential treatment towards one or the other player, that we propose a generalization of Nash's solution scheme. In this paper, we concentrate on the existence of asymmetric solution schemes, and motivate results for a similar existence theorem in the case of variable reference points.

2. <u>The Model</u>:- The class of games studied here is characterized by a pair (H, d) and by the rule that the players will attain any single payoff point in H that they jointly agree on. In the absence of an agreement, they attain d. Definitions 1 and 2 formally describe these games, Definition 3 defines the meaning of solution, and Conditions 1 to 4 define the (α_1, α_2) Nash solution to the class of games specified in Definition 1 and 2. It is a generalization of the bargaining scheme proposed in Nash (1950).

<u>Definition 1</u>:- The pair $\Gamma = (H, d)$ is a two-~~fixed~~ person fixed threat bargaining game if $H \subseteq \mathbb{R}^2$ is compact and convex, $d \in H$, and H contains at least one element, u, such that $u \gg d$.

<u>Definition 2</u>:- The <u>set of two-person fixed threat bargaining games is</u> denoted W.

The utility axioms of Von Neumann and Morgenstern are assumed throughout this paper. Consequently, one could have a bargaining game in which there were a nonconvex set (including a finite set) of outcomes. The set H would consist of the utility pairs associated with all possible lotteries defined on elements from the original set of outcomes. Such a set H would be convex. It would also be compact if the set of original outcomes were compact.

A solution to some game $\Gamma = (H, d)$ is a particular element of H which is the payoff pertaining to the solution concept under discussion, for example, the Nash solution. Because (H, d) can be any game drawn from a large set of games, a particular solution concept can be conveniently described as a function of the game, $f(H,d) \in H$.

Definition 3:- A solution to $(H, d) \in W$ is a function $f: W \longrightarrow H$, that associates a unique element of H with the game $(H, d) \in W$.

$f(H,d) = (f_1(H,d),\ f_2(H,d))$.

We now proceed to state conditions which define the (α_1, α_2) Nash Solution. Let $\alpha_1 \geqslant 0$, $\alpha_2 \geqslant 0$ be given parameters.

Condition 1:- $f(H,d) \geqslant d$ for all $(H,d) \in W$.

Condition 2:- Let $a_i \in \mathbb{R}_{++}$, and (H,d), $(H', d') \in W$ and define

$$H' = \left\{x \in \mathbb{R}^2 / (x_i - d_i')^{\alpha_i} = a_i (y_i - d_i)^{\alpha_i},\ i = 1, 2,\ y \in H\right\}.$$ Then

$$\left[f_i(H', d') - d_i'\right]^{\alpha_i} = \text{[illegible]} \left[f_i(H,d) - d_i\right]^{\alpha_i},\ i = 1, 2; \alpha_i > 0.$$

Condition 3:- If for $(H,d) \in W$, $(x_1, x_2) \in H$ implies

$$((x_2 - d_2)^{\alpha_2}, (x_1 - d_1)^{\alpha_1}) \in \left\{((y_1 - d_1)^{\alpha_1}, (y_2 - d_2)^{\alpha_2} / (y_1, y_2) \in H\right\},$$ then

$$\left[f_1(H,d) - d_1\right]^{\alpha_1} = \left[f_2(H,d) - d_2\right]^{\alpha_2},\ \alpha_i > 0.$$

Condition 4:- If $(H,d), (H',d') \in W$, $d = d'$, $H \subseteq H'$, and $f(H',d') \in H$, then $f(H,d) = f(H',d')$.

Lemma 1:- Let $(H,d) \in W$ and let $u^* = f(H,d)$ satisfy conditions 1 to 4. Then if $u \in H$ and $u \neq u^*$ either $u_1^* > u_1$ or $u_2^* > u_2$.

Proof:- Without loss of generality, we may let $d = 0$. Assume the lemma false; then there is some $y^* \in H$ such that $y^* > u^*$. Define a game $(H', 0)$ by letting

$$H' = \left\{x \in \mathbb{R}^2 / x_1^{\alpha_1} = \frac{u_1^{*\alpha_1}}{y_1^{*\alpha_1}} y_1^{\alpha_1},\ x_2^{\alpha_2} = \frac{u_2^{*\alpha_2}}{y_2^{*\alpha_2}} y_2,\ y \in H\right\} \cap H \quad (5.1)$$

Clearly, $H' \subseteq H$, $H' \neq H$, and $u^* \in H'$, because $y^* \in H$. By Condition 4, u^* is the solution to $(H', 0)$, but by Condition 2, $[(u_1^*/y_1^*)u_1^*, (u_2^*/y_2^*)u_2^*] \neq u^*$ is the solution to $(H', 0)$. This contradiction implies that the solution of $(H, 0)$ must be Pareto optimal.

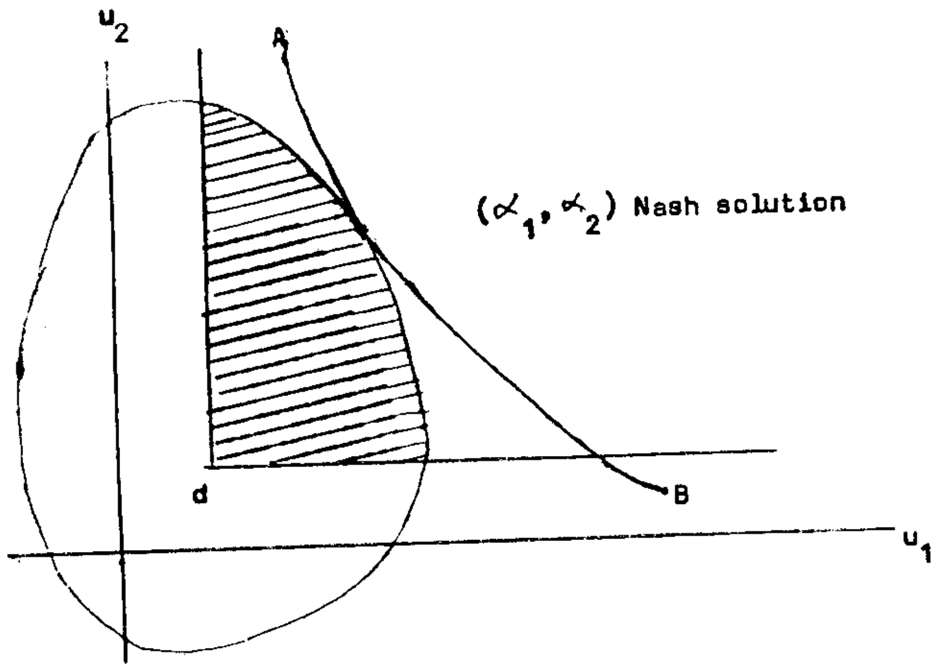
Q.E.D.

This lemma is similar in content and style to the one in Roth(1977) for $\alpha_1 = 1$, $\alpha_2 = 1$. However the conditions invoked for our proof are different, from those used by Roth. This lemma shows that under Conditions 1 to 4, $f(H, d)$ satisfies Pareto optimality.

3. Characterization and existence of the (α_1, α_2) Nash solution:-

The (α_1, α_2) Nash Solution can be characterized in a very simple way:

The element of H that maximizes $(u_1-d_1)^{\alpha 1} (u_2-d_2)^{\alpha 2}$, is the unique outcome satisfying conditions 1 to 4. This is proved in Theorem 1 and the (α_1, α_2) Nash solution is illustrated below.



The shaded area of H is the region of outcomes that are individually rational. A hyperbola, that is asymptotic to the broken lines through d is a curve along which the product $(u_1-d_1)^{\alpha 1}(u_2-d_2)^{\alpha 2}$ is constant; therefore, the point of H at which the product of gains raised to a definite power, from agreement is maximized is the point of tangency between the upper boundary of H and the most highly placed hyperbola that still touches H. The result is proved in two steps. Lemma 2 establishes the result for games with threat points at the origin (d = 0), and Theorem 1 generalizes the result to games with arbitrary threat points.

Lemma 2:- Any game $(H,o) \in W$ has a unique Nash solution $u^* = f(H,o)$ satisfying Conditions 1 to 4. The solution u^* satisfies conditions 1 to 4 if and only if

$$u_1^{*\alpha_1} u_2^{*\alpha_2} > u_1^{\alpha_1} u_2^{\alpha_2} \qquad (2)$$

for all $u \in H$, $u \geq o$, and $u \neq u^*$.

Proof:- It is first proved that if $u_1^{*\alpha_1} u_2^{*\alpha_2} = \max_{u \in H, u \geq o} u_1^{\alpha_1} u_2^{\alpha_2}$, then u^* is unique and satisfies conditions 1 to 4. Then it is shown that if $u_1^{1\alpha_1} u_2^{1\alpha_2} < u_1^{*\alpha_1} u_2^{*\alpha_2}$, u^1 violates one of the four conditions.

Suppose the function $(u_1, u_2) \longmapsto u_1^{\alpha_1} u_2^{\alpha_2}$ is maximized at (u_1, u_2).

Assuming the utility allocations to be positive, we get that the maximizer of $\alpha_1 \log u_1 + \alpha_2 \log u_2$ occurs at (u_1, u_2).

For $\alpha_1 > 0, \alpha_2 > 0$, the function $(u_1, u_2) \longmapsto \alpha_1 \log u_1 + \alpha_2 \log u_2$ is strictly concave.

Thus if for $\delta > 0, \gamma > 0$, $\alpha_1 \log (u_1 + \delta) + \alpha_2 \log (u_2 - \gamma) = \alpha_1 \log u_1 + \alpha_2 \log u_2$, then by strict concavity $\alpha_1 \log(u_1 + \frac{\delta}{2}) + \alpha_2 \log(u_2 - \frac{\gamma}{2}) > \alpha_1 \log u_1 + \alpha_2 \log u_2$. Observe that by the convexity of H, $(u_1, u_2) \in H$ and $(u_1 + \delta, u_2 - \gamma) \in H$ implies $(u_1 + \frac{\delta}{2}, u_2 - \frac{\gamma}{2}) \in H$. This contradicts that $\alpha_1 \log u_1 + \alpha_2 \log u_2$ is maximized at (u_1, u_2). Therefore the maximizer of $u_1^{\alpha_1} u_2^{\alpha_2}$, $(u_1, u_2) \in H$, $u_1 \geq 0$, $u_2 \geq$ is unique.

Turning now to Conditions 1 to 4, Condition 1 is satisfied by definition.

For Condition 2, let (H', d') be defined by $H' = \left\{x \in \mathbb{R}^2 / (x_i - d_i')^{\alpha_i} = d_i (y_i)^{\alpha_i},\ i = 1, 2,\ y \in H\right\}$. Let W be an arbitrary element of H', let u be the point in H that transforms into W, and let $W^* \in H'$ be the point to which u^*, the maximizer of $u_1^{\alpha_1} u_2^{\alpha_2}$ for(H,d) transforms. Comparing W^* and W,

$$\alpha_1 \log(W_1^* - d_1') + \alpha_2 \log(W_2^* - d_2') - \alpha_1 \log(W_1 - d_1') - \alpha_2 \log(W_2 - d_2')$$

$$= \alpha_1 \log u_1^* + \alpha_2 \log u_2^* - \alpha_1 \log u_1 - \alpha_2 \log u_2 > 0. \qquad (5)$$

To see that Condition 3 is satisfied, suppose the set $\left\{(x_1^{\alpha_1}, x_2^{\alpha_2}) / (x_1, x_2) \in H\right\}$ is symmetric about a 45° line through the origin and let $(u_1'^{\alpha_1}, u_2'^{\alpha_2}) \in \left\{(x_1^{\alpha_1}, x_2^{\alpha_2}) / (x_1, x_2) \in H\right\}$ be the point in H that is on the 45° line where it intersects the upper right boundary of H. Then $u_1'^{\alpha_1} = u_2'^{\alpha_2}$ and for any $(u_1^{\alpha_1}, u_2^{\alpha_2}) \in \left\{(x_1^{\alpha_1}, x_2^{\alpha_2}) / (x_1, x_2) \in H\right\}$, $u_1^{\alpha_1} + u_2^{\alpha_2} \leq u_1'^{\alpha_1} + u_2'^{\alpha_2}$. That is, the set $\left\{(x_1^{\alpha_1}, x_2^{\alpha_2}) / (x_1, x_2) \in H\right\}$ is bounded above by a line of slope − 1 passing through $(u_1'^{\alpha_1}, u_2'^{\alpha_2})$. The product $u_1'^{\alpha_1} u_2'^{\alpha_2}$ is strictly larger than any product of the form $u_1^{\alpha_1} u_2^{\alpha_2}$ with $u_1^{\alpha_1} + u_2^{\alpha_2} \leq u_1'^{\alpha_1} + u_2'^{\alpha_2}$; hence, $u_1'^{\alpha_1} u_2'^{\alpha_2}$ has a larger product than $u_1^{\alpha_1} u_2^{\alpha_2}$ for $(u_1, u_2) \in H$. Thus $u^* = u'$.

That condition, 4 is satisfied is trivial; for if u^* is the maximizer in H, and H is derived from H by passing away some regions of H, then $u_1^{*\alpha_1} u_2^{*\alpha_2}$ must exceed a similar product for any point in H .

A constructive argument is used to prove that if u is not the weighted payoff product maximizer, it violates at least one of the four conditions. First, let u* be the weighted payoff product maximizer for the game (H,o). Transform (H,o) into the game (H', o) by letting $H' = \left\{ y \in R^2 / y_1 = \frac{u_2^* x_1}{u_1^*} \text{ and } y_2 = x_2 \text{ for } (x_1, x_2) \in H \right\}$. Thus, the weighted product maximizer in (H', o) is (u_2^*, u_2^*). From (H', o), we obtain a third game (H'', o) by adding points to H':

Let

$$H'' = \left\{ y \in R^2 / y_1^{\alpha 1} + y_2^{\alpha 2} \leq u_2^{*\alpha 1} + u_2^{*\alpha 2} \text{ and } \left| y_1^{\alpha 1} \right| + \left| y_2^{\alpha 2} \right| \leq \max_{u \in H'} \left(\left| u_1^{\alpha 1} \right| + \left| u_2^{\alpha 2} \right| \right) \right\} \tag{6}$$

The upper right boundary of $\left\{ (y_1^{\alpha 1}, y_2^{\alpha 2}) / (y_1, y_2) \in H'' \right\}$ is a line of slope -1 through $(u_2^{*\alpha 1}, u_2^{*\alpha 2})$ and H'' contains H'; therefore, by Condition 3, (u_2^*, u_2^*) is the solution of (H'', o); by Condition 4, independence of irrelevant alternatives, (u_2^*, u_2^*) is also the solution of (H', o); and by Condition 2, $u^* = (u_1^*, u_2^*)$ is the solution of (H, o). Thus u fails to satisfy condition 1 to 4 and only u^*, the weighted product maximizer, can be the solution of the game.

Q.E.D.

Characterization of (α_1, α_2) Nash Solutions with respect to a reference point.

In the game (H, d), the threat point, d, plays the role of reference point for the nash solution. The concept of reference point can be formulated abstractly following Thomson (1981). Let g(H,d), be an element of H that is not in the Pareto set of H (ie. there is some y H such that $y \gg g(H,d)$). The reference point is required to obey two conditions that are obviously satisfied by d. With two other conditions imposed on the solution scheme, Thomson shows that the [redacted] payoff product[redacted] maximizer

$$\max_{(u_1,u_2)\in H} \left((u_1-g_1(H,d))(u_2-g_2(H,d))\right)$$

is the only candidate available which satisfies the required conditions.

Our result can be easily extended to cover solutions with respect to reference point. Conditions analogous to those suggested in Thomson(1981), paves the way for an extension of the above result.

FOOT NOTE

Max Woodbury had suggested a modification of the Nash Solution along the above lines, which takes into account different bargaining abilities.(See, Martin Shubik(1960) : "Strategy And Market Structure; Competition, Oligopoly, and the Theory of Games," John Wiley & Sons, Inc., New York).

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