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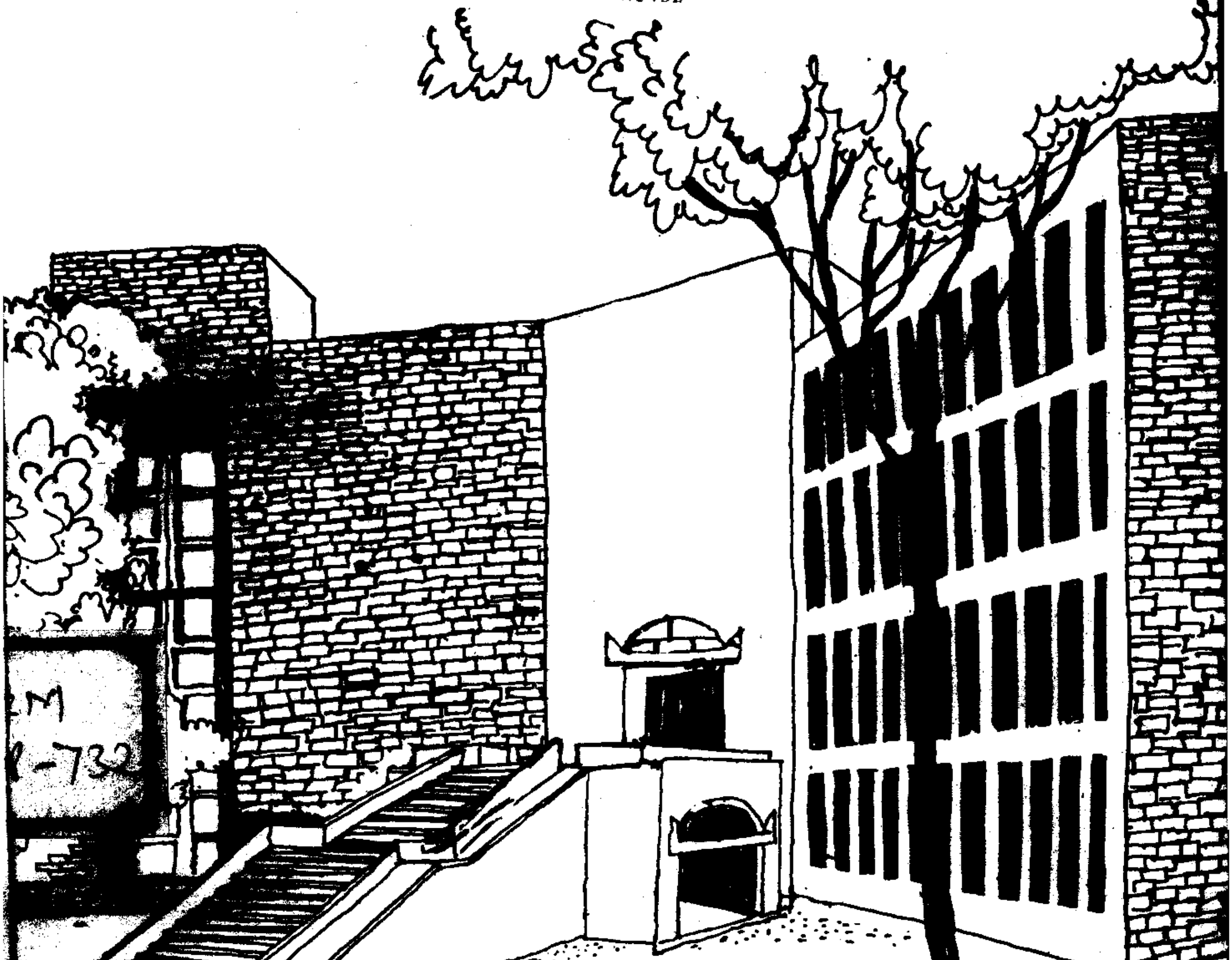

A GENERAL SADDLE POINT PROPERTY FOR TWO PERSON VARIABLE THREAT GAMES

By

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ABSTRACT

In this paper we prove that all two person variable threat games which satisfy some very general conditions, fulfil a certain saddle point property at an equilibrium point.

1. <u>Introduction</u>:- In a <u>pure bargaining problem</u> between a group of two participants there is a set of feasible outcomes, any one of which will be the result if it is specified by the unanimous agreement of all the participants. In the event that no unanimous agreement is reached, a given disagreement outcome obtains. Thus, a <u>2-person bargaining</u> problem is a pair (H, d) of a Subset H of $\mathbb{R}^2$ and of a point $d \in H$. $\mathbb{R}^2$ is the <u>utility space</u>, H is the <u>feasible set</u>, and d is the <u>disagreement point</u>. If the agents unanimously agree on a point x of H, they obtain x. Otherwise, they obtain d. Given a class of 2-person bargaining problems, a <u>solution</u> is a function F associating with every (H, d) in the class a point $F(H,d) \in H$, representing the compromise reached by the agents. In some contexts, F(H,d) may alternatively be interpreted as the compromise recommended to the agents by some impartial arbitrator.

In this paper we consider two person games in which binding agreements are possible, but in which each player has considerable scope for action in the absence of an agreement, and in which the decision of each player affects both of them. Nash (1953) dealt with this class of games from several angles. We view the above situation as in Owen (1982), which combines arbitration with a non-co-operative game. Suppose each of the two players has a strategy set, S_i for player i, that is Compact and Convex, and assume that $P_i(s_1, s_2)$ is the pay off function for player i in the absence of an agreement. Thus the game ($\{1,2\}$, S,P) is a default non-co-operative game that the two players must play if they cannot agree. There is no threat point as such. Suppose further that $H \subseteq \mathbb{R}^2$ consists of all pay off points that the two players can reach by means of binding agreements. H would naturally contain as a subset all those points attainable in the default game. Such games are called <u>variable threat games</u>. Then ($\{1,2\}$, S, F (H,P(.)), defines the associated non-co-operative game which combines both the default non-co-operative game as well as the two-person bargaining problem.

"A serious objection can be raised to Nash's bargaining scheme, and it is that it does not take threats into account. Some analysis of threats is in order if we are to correct this weakness. Quite generally, a threat is effective if it is believable, and if it tends to improve the position of the threatener vis-a-vis the person being threatened. Thus a threat to kill someone is generally more effective than a threat to become angry, because the position of the killer is certainly improved vis-a-vis his viction while getting angry generally does no such thing. On the other hand, a threat to destroy the whole world, while it may possibly improve the threatener's position in regard to others (reducing them all to equality in nothingness), is not very believable and hence not effective."

In this paper we propose a saddle point property that such equilibrium threat strategies must satisfy, under a set of very plausible assumptions.

2. <u>The Model</u>:- The class of games studied in this paper is based on a pair (H,d) and by the rule that the players will attain any single payoff point in H that they jointly agree on. In the absence of an agreement, they attain 'd'.

<u>Definition 1</u>:- The pair $\Gamma = (H,d)$ is a two-person fixed threat bargaining game if $H \subseteq \mathbb{R}^2$ is compact, convex with nonempty interior, $d \in H$, and H contains atleast one element U, such that $U \gg d$.

The requirement that H has nonempty interior, precludes the possibility that H is merely a negatively sloped line segment. In the latter case our subsequent concept, that of a solution is not well defined.

We make the following blanket assumption on H:

<u>Assumption 1</u>: If $x, y \in H$, $x \neq y$, $y \leq x$, then $tx + (1-t)y$ belong to int. (H) $\forall\, t \in (0,1)$.

Note that Assumption 1 is strictly weaker than assuming that H is strictly convex. We could without any further damage require Assumption 1 to hold only for all $x \in H$ for which there does not exist u belonging to H with $u_1 > x_1$ and $u_2 > x_2$ where $u = (u_1, u_2)$ and $x = (x_1, x_2)$.

<u>Definition 2</u>:- The <u>set of two-person fixed threat bargaining games</u> is denoted W.

<u>Definition 3</u>:- A solution to $(H,d) \in W$ is a point $F(H,d) \in H$. A solution is defined $\forall\, (H,d) \in W$.

Given $(H,d) \in W$, its Nash (1950) Solution outcome $N(H,d)$ is the point where the product $(x_1-d_1)(x_2-d_2)$ is maximized for $x = (x_1, x_2) \in H$, with

$x \geqq d$, its Egalitarian (Kalai(1977)) Solution outcome is the point $E(H,d)$ which is maximal along the ray $x_1 - d_1 = x_2 - d_2$; its Kalai-Smorodinsky(1975) Solution outcome $K(H,d)$ is the maximal point of H on the segment connecting d to $M(H,d)$, where for each i, $M_i(H,d) = \max \{ x_i / x \in H; x \geqq d \}$.

Suppose each of the two players has a strategy set, S_i for player i, that is compact and convex, and assume that $P_i(s_1, s_2)$ is the pay-off for player i in the absence of an agreement. Thus the game ($\{1,2\}$, S,P) is a default non-co-operative game that the two players must play if they cannot agree. There is no threat point as such.

Definition 4:- Γ = (N,S,P,H) is a variable threat two-person co-operative game where (N,S,P) is a two-person non-co-operative game that is played if no agreement is reached and the compact, convex set H is the co-operative attainable pay-off set. H contains $\{P(s) \in \mathbb{R}^2 / s \in S\}$.

The arbitration game proceeds by having the two players simultaneously choose strategies for the default game, $s_i \in S_i$, which are used to determine a threat point, $P(s)$. The arbitrated outcome to the game is the bargaining solution $F(H,P(s))$. Thus, the players are not interested in the payoffs $P_i(s)$ for their own sakes; they care about the effect on the final outcome that is due to their choices of S_i. Poutentially, any point on the Pareto optimal frontier of H could be an arbitrated outcome. We shall now define an equilibrium threat strategy.

Definition 5:- $s^* \in S$ is called an equilibrium threat strategy for the variable threat game (N, S, P, H) equipped with the solution

F if $F_1(H,P(s^*)) \geqslant F_1(H,P(s_1,s_2^*)) \;\forall\; s_1 \in S_1$ and $F_2(H,P(s^*)) \geqslant F_2(H,P(s_1^*,s_2)) \;\forall\; s_2 \in S_2$.

3. A General Property of Equilibrium Threat Strategies:

In this section we will establish a general property of equilibrium threat strategies for veriable threat games. Let us impose the following three conditions on F:

Condition 1:– $F(H,d) \geqslant d \;\forall\; (H,d) \in W$

Condition 2:– Let $(H,d) \in W$ and $u \in H$ with $u \neq F(H,d)$. Then either $F_1(H,d) > u_1$ or $F_2(H,d) > u_2$.

Condition 3:– Let (H,d) and $(H,d') \in W$ with $d_i' = d_i$ and $d_j' \geqq d_j$ for $j \neq i$. Then, $F_j(H,d') \geqq F_j(H,d)$. If in addition $d_j' > d_j$, then $F_j(H,d') > F_j(H,d)$.

We also need to introduce an anxilliary concept, that of a fiber (see Brito, Buoncristiani and Intriligator (1977)).

Definition 6: The set $T_i(H,d;F;u)$, $i = 1,2$, is the set of all threat points d', leading to a non-conflict allocation atleast as beneficial to player i as the Pareto optimal allocation u.

$$T_i(H,d;F;u) = \left\{ d'/d' \in H,\; F_i(H,d') \geqslant u_i \right\} \qquad (i = 1,2).$$

These sets are nested in the following sense: if $\bar{u}_i > \hat{u}_i$, then $T_i(H,d;F;\bar{u}) \subseteq T_i(H,d;F;\hat{u})$.

Definition 7:- The intersection of $T_1(H,d;F;u)$ and $T_2(H,d;F;u)$ is the set of all threat points leading to the non-conflict allocation u.

$\Pi(H,d;F;u) = T_1(H,d;F;u) \cap T_2(H,d;F;u)$ called the u-fiber.

Since it will not cause ambiguity the fiber will be denoted by $\Pi(H;d;F(H,d))$ or simply $\Pi(H,F(H,s))$.

Condition 4:- Let $(H,d) \in W$ and u be a [redacted] Pareto - optimal point of H. Then $\Pi(H,d;F,u)$ is a convex set containing more than one point. Such fibers are called Nash fibers.

Under the above conditions we can prove the following theorem.

Theorem 1:- $\Pi(H,d;F;u)$ is a positively sloped straightline. If $d \neq d'$, then $\Pi(H,d;F(H,d)) \cap \Pi(H,d';F(H,d')) = \emptyset$

Proof:- If for $d \neq d'$, $\Pi(H,d;F(H,d)) \cap \Pi(H,d';F(H,d'))$ $\neq \emptyset$, then the solution is not well defined. This takes care of the second assertion.

Now, suppose $\Pi(H,d;F(H,d))$)contains three non-colinear points. Let u_1, u_2 and u_3 be any such three points in H. Consider the triangle formed by u_1, u_2 and u_3, and the projection of the vertices onto the axes. Clearly atleast two such projections intersect two different sides of the triangle. But this would violate Condition 3. Hence $\Pi(H,d;F(H,d))$ is a straight line $\forall$ d.

Now suppose that for some d, $\Pi(H,d;F(H,d))$ is negatively sloped. This would once again contradict Condition 3 in a very obvious way. This proves the theorem.

Q.E.D.

Let us further impose the condition.

<u>Condition 5</u>:- $F(H,\cdot) : H^{\circ} \longrightarrow H$ is continuous where

$$H^{\circ} = \{ d \in H / \exists \; u \in H \text{ with } u \gg d \}$$

As a consequence of this condition we obtain,

<u>Theorem 2</u>:- The Nash fiber $\Pi(H,d;F(H,d))$ is an interval with both endpoints lying on the boundary of H.

<u>Proof</u>:- Let $A \subseteq H$, denote the set of Pareto optimal points of H. Clearly $F(H, \cdot)$ is a function from H° into A.

Let $[\bar{d}, \bar{\bar{d}}] = \Pi(H,d;F(H,d))$, where either $\bar{d}$ or $\bar{\bar{d}}$ does not belong to the boundary of H. The fact that $\Pi(H,d;F(H,d))$ is a closed interval follows from the continuity of H.

Since a continuous function maps connected sets into connected sets, we would require that the continuous image of $H^{\circ} \setminus [\bar{d}, \bar{\bar{d}}]$, which is a connected set, be connected. However $A \setminus \{F(H,d)\}$ consists of two disconnected components. Hence, we have a contradiction. So, $\bar{d}, \bar{\bar{d}} \in \partial H$ (ie. boundary of H). This proves the theorem.

Q.E.D.

Before we proceed with our next theorem let us explain the Conditions we have imposed so far. Condition 1 imposes individual rationality on the solution F. Condition 2 requires Pareto optimality of the solution and Condition 3 requires the solution to be monotone with respect to the disagreement point (See Thomson (1987)). Condition 4 requires that the set of threat points which lead to a particular arbitrated solution be a convex set. Condition 5 requires that the solution itself be continuous with respect to the disagreement point. Theorem 2, which we have proved above shows that under the above conditions the Nash fiber will be an interval whose boundaries lie on the boundary of H. Note we have not established that one of the boundary points of the fiber is F (H,d).

To establish the theorem we have in mind we also need to assume:

<u>Condition 6</u>:- $\forall$ (H,d) $\in$ W, F(H,d) is an endpoint of Π(H,d;F(H,d)).

Note, that this condition neither implies nor is implied by any of the theorems above. In general it is possible that F(H,d) is an endpoint of Π(H,d;F(H,d)) without the latter being a Nash fiber, It is also possible for a Nash fiber to have some other endpoint. Nor is the fact that Π(H,d;f(H,d)) has F(H,d) as one of its endpoints, sufficient reason to assert that <u>both</u> its endpoints lie on the boundary of H.

Now we are in a position to assert a general property of optimal threat strategies which has been shown to hold in the specific case when F is the Nash bargaining solution. (See Owen (1982)).

<u>Theorem 3</u>:- Let s* $\in$ S be an <u>equilibrium</u> <u>threat</u> <u>strategy</u> for the variable threat game (N,S,P,H) equipped with the solution F. Suppose F satisfies Conditions 1 to 6. Then,

$$\frac{F_2(H_2P(s_1^*, s_2^*)) - P_2(s_1^*, s_2)}{F_1(H,P(s_1^*,s_2^*)) - P_1(s_1^*, s_2)} \geqslant \frac{F_2(H,P(s_1^*, s_2^*)) - P_2(s_1^*,s_2^*)}{F_1(H,P(s_1^*,s_2^*)) - P_1(s_1^*,s_2^*)}$$

$$\geqslant \frac{F_2(H,P(s_1^*,s_2^*)) - P_2(s_1,s_2^*)}{F_1(H,P(s_1^*,s_2^*)) - P_2(s_1,s_2^*)}$$

$\forall (s_1, s_2) \in (S_1 \times S_2)$.

__Proof:-__ Since $s^* \in S$ is an equilibrium threat strategy

$F_2(H,P(s_1^*,s_2^*)) \geqslant F_2(H,P(s_1^*,s_2)) \; \forall \; s_2 \in S_2$ and

$F_1(H,P(s_1^*,s_2^*)) \geqslant F_1(H,P(s_1,s_2^*)) \; \forall \; s_1 \in S_1$.

Suppose $\Pi(H,P(s_1^*,s_2))$ lies above $\Pi(H,P(s_1^*,s_2^*))$. Consider the projection of any point $d \in \Pi(H,P(s_1^*, s_2))$ onto the payoff axis of player 2. It intersects $\Pi(H,P(s_1^*, s_2^*))$ at a pt. $d' = (d_1', d_2')$ where $d_1' = d_1$ and $d_2' < d_2$. By monotonicity,

$F_2(H,P(s_1^*,s_2^*) = F_2(H,d') < F_2(H,d) = F_2(H,P(s_1^*, s_2))$ contradicting s^* is an equilibrium threat strategy. Hence $\Pi(H,P(s_1^*, s_2))$ lies on or below $\Pi(H,P(s_1^*, s_2^*)) \; \forall \; s_2 \in S_2$. By a symmetric argument $\Pi(H,P(s_1,s_2^*))$ lies on or above $\Pi(H,P(s_1^*,s_2^*)) \; \forall \; s_1 \in S_1$.

Since $P(s_1^*, s_2) \in \Pi(H,P(s_1^*,s_2))$ for $s_2 \in S_2$, for $s_2 \neq s_2^*$, the slope of the line joining $P(s_1^*,s_2)$ to $F(H,P(s_1^*,s_2^*))$ is greater than or equal to the slope of the line joing $P(s_1^*, s_2^*)$ to $F(H,P(s_1^*, s_2^*))$. This proves the first inequality. The second inequality is established analogously and this proves the theorem.

Q.E.D.

Our next theorem is a mere restatement of the fact that (N,S,P,H) equipped with F is a competitive game if Condition 2 is satisfied.

Theorem 4:- Let (N,S,P,H) equipped with F satisfy Condition 2. Then if $\bar{s} \in S$ and $s^* \in S$ are equilibrium threat strategies,

$$F(H,P(s^*) = F(H,P(\bar{s})).$$

Proof:- Suppose towards a contradiction, $F(H,P(s^*)) \neq F(H,P(\bar{s}))$. Without loss of generality assume that $F_2(H,P(s^*)) > F_2(H,P(\bar{s}))$. By Condition 2, $F_1(H,P(s^*)) < F_1(H,P(\bar{s}))$.

But $F_1(H,P(s^*)) \geqslant F_1(H,P(s_2^*,\bar{s}_1))$ implies, $F_2(H,P(s_2^*,\bar{s}_1)) \geqslant F_2(H,P(s^*))$ by Condition 2.

However, $F_2(H,P(s_2^*, \bar{s}_1) \leqslant F_2(H,P(\bar{s}))$.

$\therefore F_2(H,P(\bar{s})) \geqslant F_2(H,P(s^*))$ which is a contradiction. This proves the theorem.

Q.E.D.

4. Conclusion:- In the above analysis we have established a general saddle point property satisfied by all equilibrium threat strategies in a variable threat game if certain conditions are satisfied. The more well know bargaining solutions, for instance, those proposed by Nash(1950), Kalai and Smorodinsky(1975), Kalai (1977) and Lahiri(1988) all satisfy the above conditions. The conditions are quite general and are usually satisfied by most bargaining solutions.

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