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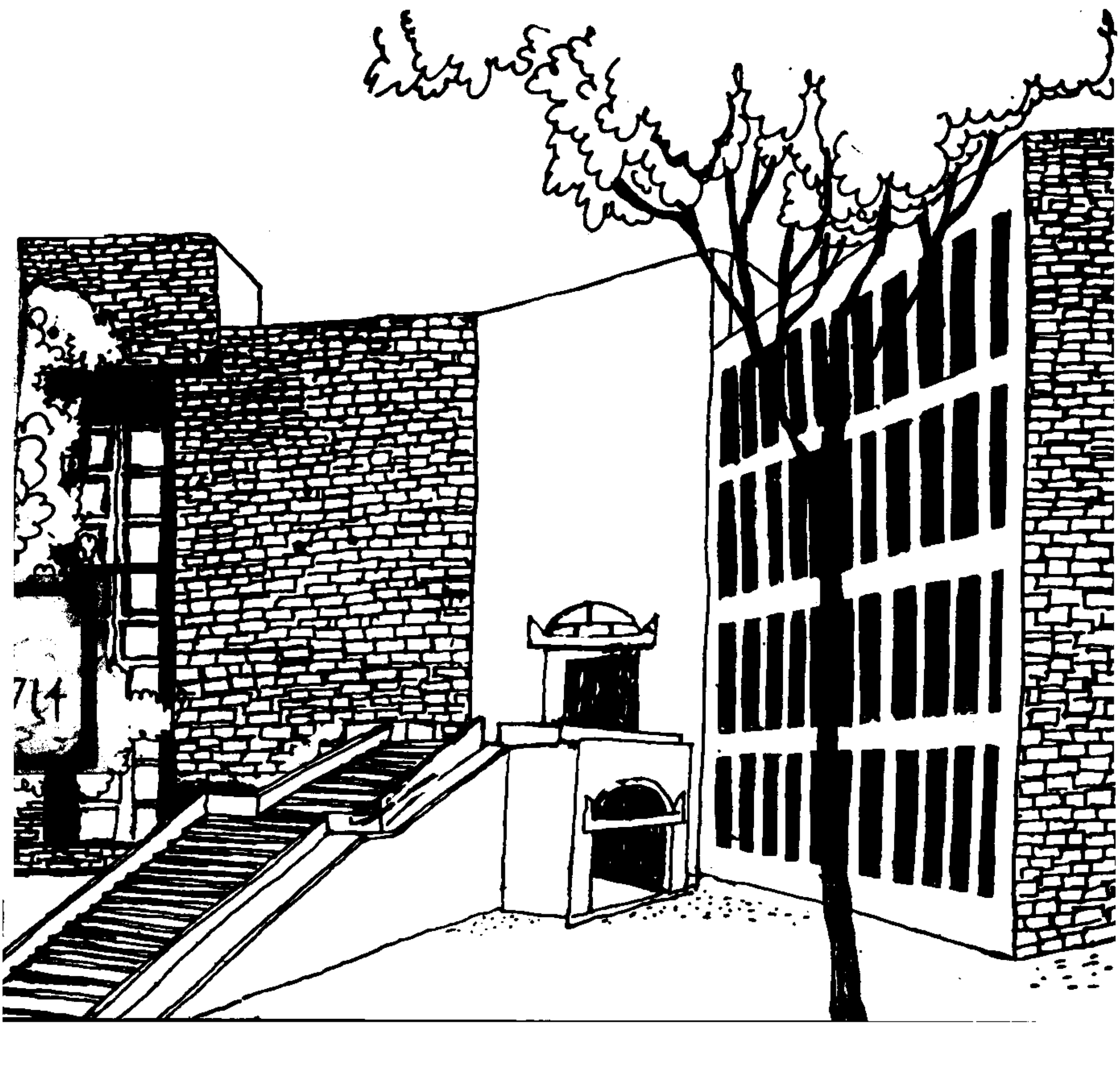

A GENERALIZATION OF THE RAIFFA - KALAI - SMORODINSKY SOLUTION IN TWO PERSON BARGAINING GAMES

By

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A B S T R A C T

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1. <u>Introduction</u>:- In this paper we present a generalization of the Raiffa - Kalai - Smorodinsky Solution to two person bargaining games, to incorporate institutional asymmetries in solution payoffs.

The solution concept proposed by Raiffa, treats both players symmetrically. However, more often than not, in practice we find departures from fairness in arbitration mechanisms. It is to incorporate, these asymmetries that we extend the above mentioned co-operative game solution.

2. The Model:- The class of games to be analysed in this paper is characterized by a pair (H, d) where $H \subseteq \mathbb{R}^2$ denotes the set of attainable payoff pairs and $d \in H$ denotes the threat point. If the players fail to achieve an agreement, they will receive the payoffs $d = (d_1, d_2)$. If a payoff point $u = (u_1, u_2) \notin H$, then it is impossible for the players to achieve it, but if $u \in H$, then there are joint actions open to them that will result in u being the payoff.

Definition 1:- The pair $\Gamma = (H,d)$ is a two-person fixed threat bargaining game if $H \subseteq \mathbb{R}^2$ is compact and convex, $d \in H$, and H contains atleast one element, u, such that $u \gg d$.

Definition 2:- The set of two-person fixed threat bargaining games is denoted W.

A solution to some game $\Gamma = (H,d)$ is a particular element of H which is the payoff pertaining to the solution concept under discussion.

Definition 3:- A solution is a function $f: W \longrightarrow \mathbb{R}^2$ that associates a unique element of H with the game $(H,d) \in W$ ie. $f(H,d) \in H$.

The most familiar solution concept is that due to Nash (1950). Several alternatives to the nash approach have been put forth. A solution concept suggested by Raiffa and described briefly in Luce and Raiffa (1957) has been axioniatized by Kalai and Smorodinsky(1975).

The basic intuitive notion is that each player naturally aspires to have the largest payoff available in the game that is consistent with individual rationality. These two individual payoffs are, in general, not attainable simultaneously and the Raiffa - Kalai - Smorodinsky Solution is to settle at the largest attainable payoff that is proportional to them.

Here we attempt a generalization of the Raiffa - Kalai - Smorodinsky (henceforth referred to as RKS) solution, to incorporate inherent asymmetries in the arbitration procedure. We call our solution concept the <u>θ - RKS solution</u>. Before we proceed further let us introduce a few concepts.

Let

$$M_1(H,d) = \max \left\{ x_1 \in \mathbb{R} / (x_1, x_2) \in H \text{ and } x_2 \geq d_2 \right\} \quad (1)$$

$$m_2(H,d) = \max \left\{ x_2 \in \mathbb{R} / (M_1(H,d), x_2) \in H \right\} \quad (2)$$

$$M_2(H,d) = \max \left\{ x_2 \in \mathbb{R} / (x_1, x_2) \in H \text{ and } x_1 \geq d_1 \right\} \quad (3)$$

$$m_1(H,d) = \max \left\{ x_1 \in \mathbb{R} / (x_1, M_2(H,d)) \in H \right\} \quad (4)$$

$M_1(H,d)$ is the largest payoff in H for player 1 among all outcomes at which player 2 obtains atleast d_2. $m_2(H,d)$ is the payoff player 2 receives when player 1 gets $M_1(H,d)$. If the payoff for player 2 corresponding to $M_1(H,d)$ is not unique, then $m_2(H,d)$ is the largest of the corresponding payoffs. $M_2(H,d)$ and $m_1(H,d)$ are analogously defined.

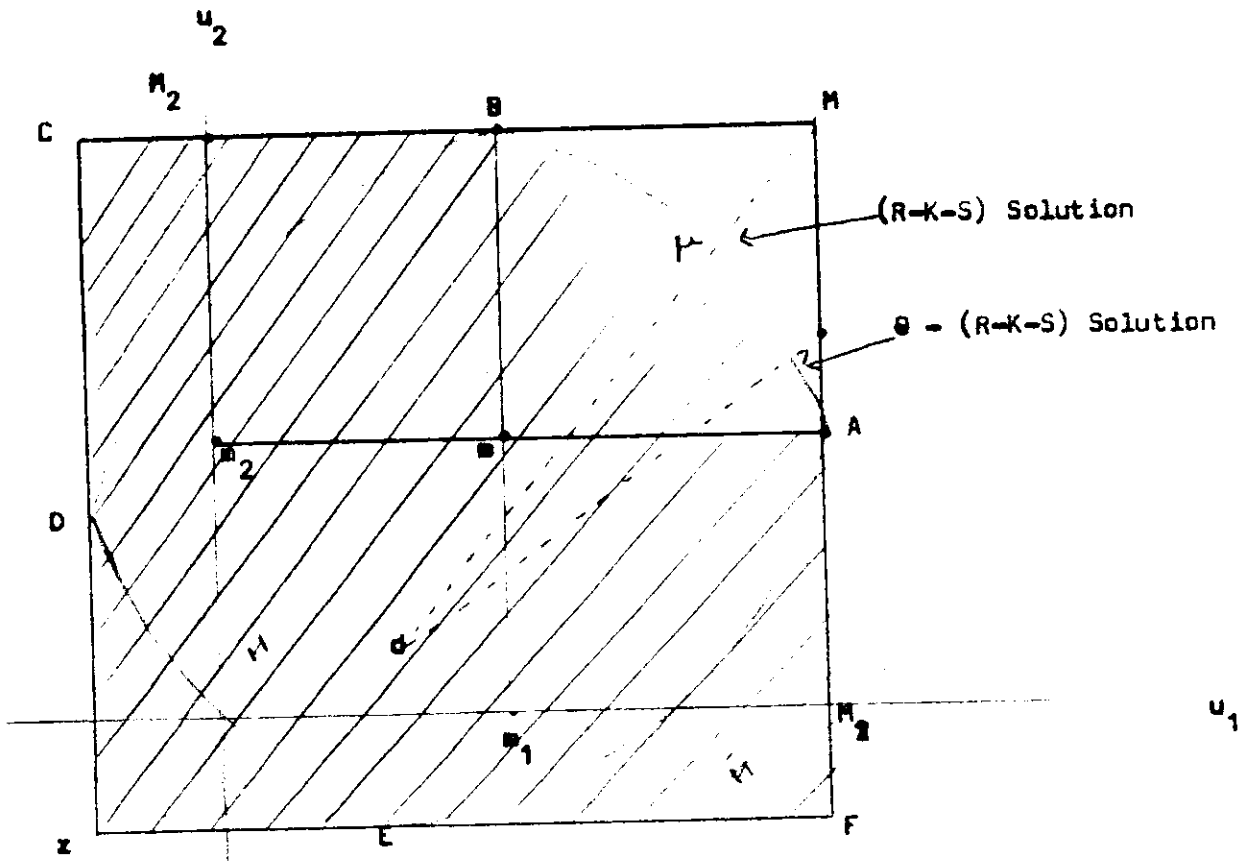
<u>Definition 4</u>:- The point M (H,d) = $(M_1(H,d), M_2(H,d)$ is called the <u>ideal point</u>.

<u>Definition 5</u>:- The point m (H,d) = $(m_1(H,d), m_2(H,d))$ is called <u>the point of minimal expectations</u>.

Note that, in general, the ideal point is not an element of H. Indeed, if it were there would be nothing to bargain over (ie. no conflict in the interests of the two players).

Let θ belong to the interval

$$\left[\frac{m_2(H,d) - d_2}{M_2(H,d)-d_2},\ \frac{M_1(H,d) - d_1}{m_1(H,d) - d_1}\right]$$



<u>Figure 1</u>

The conditions defining the θ - (R-K-S) solution are:

<u>Condition 1</u>:- $f(H,d) \geqslant d$ for all $(H,d) \in W$

<u>Condition 2</u>:- Let $a_1, a_2 \in \mathbb{R}_{++}$, $b_1, b_2 \in \mathbb{R}$, and (H,d), $(H', d') \in W$ and define $d_i' = a_i d_i + b_i$, $i = 1, 2$, and $H' = \{x \in \mathbb{R}^2 / x_i = d_i y_i + b_i,\ i = 1,2, y \in H\}$. Then $f_i(H', d') = d_i f_i(H,d) + b_i$, $i = 1, 2$.

<u>Condition 3</u>:- If $(H,d) \in W$ satisfies $d_1 = d_2$ and $(x_1, x_2) \in H$ implies $(x_2, x_1) \in H$, Then $f_2(H,d) = \theta\, f_1(H,d) + (1-\theta)\, d_1$

We now add a <u>free disposal of utility</u> assumption to the model. To that end, let $z_i = \min. \{y_i \in \mathbb{R} / (y_1, y_2) \in H\}$ for $i = 1,2$. Thus z_i is the smallest payoff in H that player i could concievably receive, and $z = (z_1, z_2)$. The set $\bar{H}$ is defined as the set of payoff points that both weakly dominate z and that are weakly dominated by an element of H. That is,

$$\bar{H} = \{y \in \mathbb{R}^2 / y \geqslant z \text{ and, for some } x \in H,\ x \geqslant y\} \qquad (5)$$

<u>Condition 4</u>:- If $x > f(H,d)$ then $x \notin H$

<u>Condition 5</u>:- Let (H,d) and (H',d') satisfy (a) $d = d'$, (b) $M_1(H,d) = M_1(H',d')$, and (c) $\bar{H} \subset \bar{H}'$. Then $f_2(H,d) \leq f_2(H',d')$.

Condition 4, <u>strong Pareto optimality</u>, states that there is no alternative in H that gives more to one player than $f(H,d)$ without giving less to the other. <u>Weak Pareto optimality</u> states that $u \in H$ is Pareto optimal if there is another point that gives strictly more to each player. Condition 6, monotonicity, states that if two games yield the same maximum payoff to player 1

$(M_1(H,d) = M_1(H',d'))$, the second game affords atleast as large a maximum payoff as the first to player 2, they have the same threat point, and the Pareto set of the second game lies on or above the Pareto set of the first game ($\bar{H} \subset \bar{H}'$), then the solution payoff to player 2 in the second game must be atleast as large as in the first game. The conditions on M_2 and the payoff frontier can be stated as stipulating that, for fixed x_1, the largest value of x_2 is atleast as large in $\bar{H}'$, and in $\bar{H}$. The point here is that the game (H', d') is atleast as favourable to player 2 as the game (H, d), while the two games retain the same value of M_1. In this situation, monotonicity requires that player 2 get at least as large a payoff from the second game as from the first.

Now let the solution $f(H,d)$ be defined by the following two conditions:

(a) $$\frac{f_2(H,d) - d_2}{f_1(H,d) - d_1} = \theta \; \frac{M_2(H,d) - d_2}{M_1(H,d) - d_1} \qquad (6)$$

(b) if

$$\frac{x_2 - d_2}{x_1 - d_1} = \theta \; \frac{M_2(H,d) - d_2}{M_1(H,d) - d_1} \qquad (7)$$

and $x > f(H,d)$, then $x \notin H$.

Condition (a) states that the ratio of the amount each player receives over and above his threat point payoff is proportional to the ratio of the amount each player receives at the ideal point over and above his threat point payoff. The proportionality constant θ measures the degree of asymmetry in the arbitration scheme.

3. Existence of the θ - (R - K - S) Solution:-

Analogous to the main result of Kalai and Smorodinsky (1975) is our result that the function $f(H,d)$ defined by (a) and (b) above, is well defined and is the only function that satisfies Conditions 1 to 5. This is proved in Theorem 1 with the aid of the following two lemmas.

Lemma 1:— A non-negatively sloped ray (ie. straight line) through d passes through a strongly Pareto optimal point of H if and only if the slope of the ray lies in the interval

$$\left[\frac{m_2(H,d) - d_2}{M_1(H,d) - d_1}, \frac{M_2(H,d) - d_2}{m_1(H,d) - d_1}\right] \qquad (8) \quad .$$

Proof:- A non-negatively sloped ray whose slope lies in the interval (8) must pass through a point on the (weak) upper right frontier of H. (A point $u \in H$ is on this frontier if there is no $u' \in H$ such that $u' \gg u$.) To fail to do so would contradict the compactness of H, and for this point to fail to be Pareto optimal would, likewise, contradict the convexity of H. A ray of slope steeper than the upper bound in (8) could only intersect the upper right boundary of H at a point where $x_1 < m_1(H,d)$ and $x_2 \leq M_2(H,d)$; therefore, x could not be strongly Pareto optimal. A parallel argument can be made for non-negatively sloped rays that are too flat to lie in the interval (8).

Q.E.D.

Lemma 2:- The function f (H,d) defined by equations (6) and (7) has a unique value for every (H,d) and corresponds to a strongly Pareto optimal element of H for θ belonging to the interval

$$\left[\frac{m_2(H,d)-d_2}{M_2(H,d)-d_2}, \frac{M_1(H,d)-d_1}{m_1(H,d)-d_1}\right] \tag{9}$$

<u>Proof</u>:- $f(H,d)$ is that point in H which lies on a ray through d of slope $\theta\left[M_2(H,d)-d_2\right] / \left[M_1(H,d)-d_1\right]$. This slope, for θ in the interval (9), lies in the interval (8) ; therefore, it passes through a strongly Pareto optimal element of H. The ray could not pass through two or more Pareto optimal elements, because its slope is positive and, hence, one of the supposed Pareto optimal points would strongly dominate the other.

Q.E.D.

<u>Theorem 2</u>:- The function $f(H,d)$ is well defined, satisfies Conditions 1 to 5, and is the only function to satisfy these conditions.

<u>Proof</u>:- First we show that $f(H,d)$ satisfies the several conditions and is well defined; then it is shown that it is the only function to satisfy the conditions. It is proved in Lemma 2 that $f(H,d)$ is well defined and satisfies Condition 4. That Condition 1 is satisfied is obvious; Condition 2 is straightforward and Condition 3 holds because for a symmetric game, $M_1(H,d) = M_2(H,d)$, $d_1 = d_2$.

To see that Condition 5 is satisfied, suppose that (H,d) and (H', d') are two games satisfying $d = d'$, $M_1(H',d')$, and $\bar{H} \subseteq \bar{H}'$. The latter implies that $M_2(H,d) \leq M_2(H', d')$. The ray from a threat, point d through a point $f(H,d)$ is referred to as the <u>definning ray</u> for $f(H,d)$. Note the following facts: (a) Because $M_1(H,d) = M_1(H', d')$ and $M_2(H,d) \leq M_2(H', d')$, the defining ray for $f(H', d')$ has a slope atleast as great as that for $f(H,d)$. (b) The set of Pareto optimal points of H is identical with that of $\bar{H}$, and the same holds for H' and $\bar{H}'$. (c) Let y denote the point on the upper right boundary of $\bar{H}$ that the defining ray of $f(H,d)$ passes through, and let $\bar{y}$ denote the point on the upper right boundary of $\bar{H}$ that the defining ray of $f(H',d')$ passes through. Then $\bar{y}_2 \geq y_2$. (d) Let y' denote the point on the upper boundary of H' that the defining ray of $f(H',d')$ passes through. Then $y'_2 \geq \bar{y}_2$. Now note that y is on the upper right boundary of H; hence $y = f(H,d)$. Similarly, y' is on the upper right boundary of H', implying that $y' = f(H',d')$. Therefore, $f_2(H',d') \geq f_2(H,d)$, which establishes that $f(H,d)$ satisfies Condition 5.

To see that only $f(H,d)$ satisfies the various Conditions, it is first shown to hold for games in which $d = 0$ and $M_1(H,0) = M_2(H,0) = 1$. Extension to the full class of games follows from the requirement that the solution satisfy invariance to affine transformations. Denote the true solution by $f^*(H,d)$, and let H' be the convex hull of the set of points $\{(0,0), (0,1), (1,0), g(\bar{H}, 0),\}$ (λ) where $g(\bar{H},0)$ is the point on the upper right boundary of $\bar{H}$ where the line through $(0,0)$ and $(1,1)$ intersects it. Monotonicity requires that $f^*(\bar{H},0) \geq f^*(H,0)$ and that $f^*(\bar{H},0) \geq f^*(H',0)$, because $H \subseteq \bar{H}$ and $H' \subseteq H$. At the same time, the definition of H and the Pareto optimality of $f^*(H,0)$ imply

$$f^*(H,0) = f^*(\bar{H},0) \tag{10}$$

Condition 3 assures that

$$\frac{f_2^*(H',0)}{f_1^*(H',0)} = \frac{f_2(\bar{H},0)}{f_1(\bar{H},0)} \tag{11}$$

Meanwhile, $f(\bar{H},0)$ is in the Pareto optimal set of both H and $\bar{H}$, hence,

$$f(\bar{H},0) = f(H,0) \tag{12}$$

and

$$\frac{f_2^*(\bar{H},0)}{f_1^*(\bar{H},0)} = \frac{f_2^*(H',0)}{f_1^*(H',0)} \tag{13}$$

Equations (11) and (13) imply

$$\frac{f_2^*(\bar{H},0)}{f_1^*(\bar{H},0)} = \frac{f_2(\bar{H},0)}{f_1(\bar{H},0)}$$

Since both f^* and f are Pareto optimal, this implies

$$f^*(\bar{H},0) = f(\bar{H},0). \tag{14}$$

Equations (10), (12) and (14) imply that $f^*(H,0) = f(H,0)$.

Q.E.D.

N.B. While establishing that the θ - R-K-S solution satisfies Condition 5, it was assumed that the defining ray of $f(H', d')$ intersects the upper right frontier of $\bar{H}$ at $\bar{y}$. Suppose that this is not the case. Then the upper right frontier of $\bar{H}$ lies completely below the defining ray of $f(H', d')$. Hence y, the point of intersection of the defining ray of $f(H, d)$ with the upper right frontier of $\bar{H}$, lies below the defining ray of $f(H', d')$. Since $\bar{H} \subseteq \bar{H}'$, the upper right frontier of $\bar{H}'$ lies to the north-east of the upper right frontier of $\bar{H}$. Let y' be the point of intersection of the defining ray of $f(H', d')$ with the upper right frontier of $\bar{H}'$. Therefore, $y'_2 \geq y_2$. Now, observing that $y' = f(H', d')$ and $y = f(H, d)$, we get that $f_2(H', d') \geq f_2(H, d)$.

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