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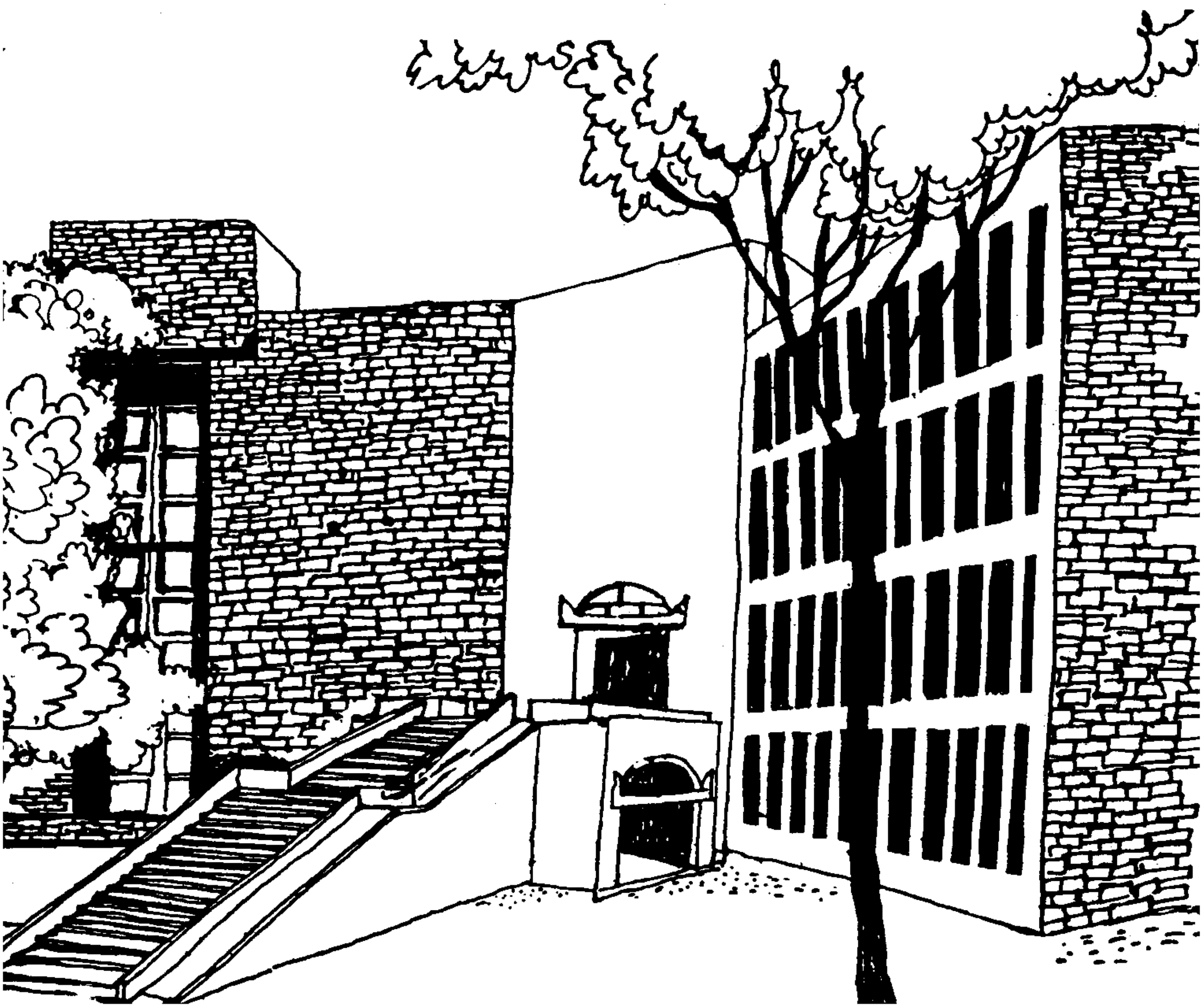

TWO CHARACTERIZATIONS OF THE INDEPENDENCE OF IRRELEVANT ALTERNATIVES ASSUMPTION

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ABSTRACT

In this paper we propose alternative characterizations of the Independence of Irrelevant Alternatives (IIA) assumption, as conceived in bargaining theory. We provide two distinct characterizations, which, allows us to view both the deficiencies as well as the advantages of the IIA axiom from a new angle.

1. Introduction : In this paper we propose alternative characterizations of the Independence of Irrelevant Alternatives (IIA) assumption, as conceived in bargaining theory. We provide two distinct characterizations, which, allows us to view both the deficiencies as well as the advantages of the IIA axiom from a new angle.

2. The Model :- An n-person bargaining problem, or simply a problem, is a pair (S,d), where S is a subset of $\mathbb{R}^n$ and d is a point in S, such that (1) S is convex and closed, (2) (S,d) is d-comprehensive, i.e. for all $x \in S$ and for all $y \in \mathbb{R}^n$, if $d \leqq y \leqq x$, then $y \in S$, and (4) there exists $x \in S$ with $x \gg d$.

S is the feasible set. Each point x of S is a feasible alternative. The coordinates of x are the utility levels, measured in some von Neumann-Morgenstern scales, attained by the n agents indexed by $i \in \{1,\ldots,n\}$ through the choice of some joint action. The point d is the disagreement point. The intended interpretation of (S,d) is as follows : the agents can achieve any point of S if they unanimously agree on it. If they do not agree on any point, they end up at d. Let Σ^n be the class of all n-person problems.

A solution is a function $F : \Sigma^n \to \mathbb{R}^n$ such that for all (S,d) Σ^n, $F(S,d) \in S$. $F(S,d)$, the value taken by the solution F when applied to the problem (S,d), is called the solution outcome of (S,d).

3. IIA and WARP :- In this section we are interested in solutions satisfying either one of the two following axioms :

Independence of Irrelevant Alternatives (IIA) : If $(S,d) \in \Sigma^n$, $(T,d) \in \Sigma^n$, $T \subsetneq S$ and $F(S,d) \in T$, then $F(T,d) = F(S,d)$.

Weak Axiom of Revealed Preference (WARP) : If $(S,d) \in \Sigma^n$, $(T,d) \in \Sigma^n$, and $F(T,d) \in S$, then $F(S,d) \notin T$ provided $F(S,d) \neq F(T,d)$.

The definition of IIA is standard. The definition of WARP is an easy extension of the conventional definition in demand analysis and says that for two different bargaining problems with the same disagreement point if the solution outcome of the second problem is feasible for the first problem, then the solution outcome for the first problem is infeasible for the second.

We now establish the main conclusion of this section :

<u>Theorem 1</u> :- $F : \Sigma^n \to \mathbb{R}^n$ satisfies IIA if and only if it satisfies WARP.

<u>Proof</u> :- (WARP) => (IIA) : Let the conditions of IIA be satisfied where $(S,d) \in \Sigma^n$, $(T,d) \in \Sigma^n$, $T \subseteq S$ and $F(S,d) \in T$. Suppose $F(T,d) \neq F(S,d)$. Then since $F(T,d) \in T \subseteq S$, by WARP, $F(S,d) \notin T$ which is a contradiction.

(IIA) => **(WARP)** : **Let $(S,d) \in \Sigma^n$ and** $(T,d) \in \Sigma^n$. **Then** by conditions (2) and (3) of the definition of a bargaining problem, $(S \cap T, d) \in \Sigma^n$. Suppose $F(T,d) \in S$, $F(T,d) \neq F(S,d)$ and towards a contradiction that $F(S,d) \in T$.

$\therefore$ $F(T,d) \in S \cap T$ implies by IIA that $F(S \cap T, d) = F(T,d)$ and $F(S,d) \in S \cap T$ implies by IIA that $F(S \cap T, d) = F(S,d)$.

However, $F(S,d) \neq F(T,d)$ leads to a contradiction and completes the proof.

<u>4. IIA</u> and <u>IIT</u> :- In this section we assume for the sake of simplicity that $n = 2$. Let $B \equiv \Sigma^2$.

For $(S,d) \in B$, we denote by P (S) the <u>Pareto Optimal Subset of S</u>:

$P(S) \equiv \{x \in S / \text{for all } y \in S, \text{ if } y \geqq x \text{ then } y = x\}$, and by P $(S,d) \equiv \{x \in P(s) : x \geqq d\}$ the <u>individually rational Pareto Optimal</u> **set** of (S,d). Further $S_d \equiv \{x \in S: x \geqq d\}$, and $W(S) \equiv \{x \in S / y = (y_1, y_2),\ y_i > x_i,\ i = 1,2 \text{ implies } y \notin S\}$.

For reasons which are primarily technical we make the following blanket assumption:

<u>Assumption</u> : - $\forall (S,d)\in B$, and $x = (x_1,x_2)\in P(S,d)$, if there exists $(p_1,p_2) = p\in R^2_+$, $p_1 + p_2 = 1$ such that $p_1x_1 + p_2x_2 \geqslant p_1y_1 + p_2y_2$, $= (y_1,y_2)\in S$, then $p_1 > 0$ and $p_2 > 0$.

Nash (1950) proposed the following properties for a solution F on B.

WPO (<u>Weak</u> <u>Pareto</u> <u>Optimality</u>) : $F(S,d)\in W(S)$ for every $(S,d)\in B$

IR (<u>Individual</u> <u>Rationality</u>) : $F_i(S,d) \geq d_i$, $i = 1,2$ for every $(S,d)\in B$

IAUT (<u>Independence</u> <u>of</u> <u>positive</u> <u>affine</u> <u>utility</u> <u>transformations</u>):

For all a, $b\in R^2$ with $a>0$ and every $(S,d)\in B$, we have $F(aS + b,\ ad + b) = a\,F(S,d) + b$. Here $ax\equiv(a_1x_1,\ a_2x_2)$ for $x\in R^2$, and $aT\equiv\{ax:\ x\in T\}$ for $T\subset R^2$.

SYM (<u>Symmetry</u>) : If $(S,d)\in B$ is <u>symmetric</u>, i.e. $d_1 = d_2$ and $S = \{(x_2,x_1) : x\in S\}$, then $F_1(S,d) = F_2(S,d)$.

IIA (<u>Independence</u> <u>of</u> <u>Irrelevant</u> <u>Alternatives</u>) : For all (S,d), $(T,d)\in B$ with $S\subset T$ and $F(T,d)\in S$, we have $F(T,d)= F(S,d)$.

Nash (1950) proved the following theorem.

<u>Theorem</u> <u>2</u> :- The Nash solution $N : B \to R^2$ is the unique solution with the properties WPO, IAUT, SYM and IIA. $\left(N(S,d) = \underset{x_i\geq d_i,\ i=1,2,\ x\in S}{\arg\max}(x_1-d_1)(x_2-d_2)\right)$

The IIA-property is the most debated property in the literature on the Nash bargaining solution (see e.g. Kalai and Smorodinsky (1975)). What the IIA property says is that if the set of underlying alternatives shrinks while the original solution alternative is still available, then the new solution alternative should be the originally available solution alternative. Two other properties for a solution f on B are defined as follows :

SIR (Strong Individual Rationality) : F (S,d)>>d for every (S,d)∈B

PO (Pareto Optimality) : F (S,d)∈P(S) for every (S,d)∈B

A bargaining problem can be viewed as a decision problem in which the decision maker consists of the two bargainers as a group, and in which the decision or compromise is the point assigned by some solution F. In this context one might expect that the "decision maker" would maximize certain "preferences:" formally, we say that the binary relation $\succsim$ on R^2 represents F if for every game (S,d) there is a unique point z with $z \succsim x$ for all x in S, and z = F(S,d). In light of this we may state as in Peters and Wakker (1987).

Theorem 3 :- There exists a binary relation $\succsim$ on R^2 representing f if and only if f satisfies IIA.

This brings out the significance of the IIA axiom. However, the unreasonableness of the IIA axiom has given rise to a spate of alternative axioms and the formulation we shall propose in this paper is one such, which at the same time highlights the simple geometry of the IIA axiom. Our formulation is motivated by the work of Shapley (1969), where he suggests a method of selecting a set of self-justifying weights for a generalized utilitarian social welfare function. This method also underlies the definition of the "modified Shapley value."

The property we suggest both as a geometric characterization of IIA as well as an alternative to it rests on the supporting hyperplane theorem (see Rockafellar (1970), Section 11), by which if <u>$F(S,d) \in B$ and $x \in P(S)$ then there exists</u> $p \in R^2_+$, $p_1 + p_2 = 1$, <u>such that $p. x \geq p.y$ for all $y \in S$. This holds since S is assumed to be compact and convex for all $(S,d) \in B$.</u>

Given $p \in R^2_{++}$, $p_1 + p_2 = 1$, $x \in R^2$ and $d \in \mathbb{R}^2$ with $p.d < p. x$, we denote $S\ (p,\ x,\ d) \equiv \{y \in R^2 / p.\ y \leq p.\ x$ and $y \geq d\}$.

Hence $(S\ (p,x,d),d)$ is a game in B.

We shall now mention the property which we propose in this section as an alternative ' to, as well as a partial characterization of IIA.

IIT (<u>Independence of Irrelevant Transfers</u>) :- Given $(S,d) \in B$, and $x \in P(S)$, if $f(S(p,x,d),d) \in S$ for some $p \in R^2_{++}$, $p_1 + p_2 = 1$, then $f(S,d) = f(S(p,x,d),d)$.

The intuition behind IIT is clear. Consider the weights $p = (p_1,p_2)$ which play the role of conversion rates that transform individual utilities into some universal unit. In particular, these weights act also as <u>rates of transformation</u> or <u>rates of exchange</u> between individual utilities. Every choice of weights $p = (p_1,\ p_2)$ defines a system of transferable utility between the individuals where the ratio of the weights determine the rate at which utility side payments between the individuals are to be made. Once utility is transferable we can construct from the given game (S,d), a simpler game $(S(p,x,d),d)$ and apply our solution concept to this game. In general the solution will not be feasible for the underlying game. In such a case $p = (p_1,p_2)$ <u>fails to justify itself</u>, in the sense that these rates of utility transfers, do not correctly reflect the realities of the underlying situation at $x \in P(S)$. Therefore, a new set of rates for utility transfer must be examined and the process must be repeated until a set of weights, say $p^* = (p_1^*, p_2^*)$, is found having the property that the solution to the associated simplified game is feasible for the game (S,d). Such weights are self justifying, and the alternatives in the simplified game which do not coincide with the solution are irrelevant from our stand point. Let us make the following assumption about our bargaining solution.

(<u>CONT</u>.) (Continuity) : The solution $F : B \to R^2$ is continuous i.e. if there exists a sequence $\{(S^k, d^k)\}_{k=1}^{\infty}$ of games belonging to B such that

$\lim_{k\to\infty} S^k = S \subset R^2$ in the Hausdorf topology and $\lim_{k\to\infty} d^k = d \in R^2$ and $(S,d) \in B$, then $\lim_{k\to\infty} F(S^k, d^k) = F(S,d)$.

It is easily established by appealing to Brouwer's fixed point theorem, that if $F: B \to R^2$ satisfies (P,O), (IR) and (CONT), then there exists $x^* \in P(S)$ $p^* \in R^2_{++}$, $p_1^* + p_2^* = 1$ such that $f(S(p,x,d),d) = x^* \in S$. In view of this we can state and prove the following main theorem of this section:

<u>Theorem 4</u> :- Let $F: B \to R^2$ be a solution satisfying PO, IR and CONT. Then IIA implies and is implied by IIT.

<u>Proof</u> :- Suppose F satisfies IIA. Then IIT is immediate.

Conversely, suppose F satisfies PO, IR, CONT and IIT.

Let (S,d) and (T,d) be two games belonging to B and $F(T,d)$ S.

We have to show that $F(S,d) = F(T,d)$.

By PO, IR and CONT., there exists $x^* \in P(T)$ and $p^* \in R^2_{++}$, $p_1^* + p_2^* = 1$ such that $x^* = F(S, p^*, x^*, d), d) \in T$

By IIT, $F(T,d) = F(S(p^*, x^*, d), d) \in T$

But $f(T,d) \in S$ implies $x^* = F(S(p^*, x^*, d), d) \in S$.

$x^* \in P(T)$ and $S \subset T$ implies $x^* \in P(S)$

Hence by IIT, $x^* = F(T,d) = F(S(p^*, x^*, d), d) = F(S,d)$

The above theorem provides a simple geometric explanation of the IIA axiom whenever a solution satisfies PO, IR and CONT. It says that consider the supporting hyperplanes to a game at each Pareto optimal point and apply the given solution to the simplified game described earlier. If for some such simplfied game the Pareto optimal point generating it coincides with the solution to the simplified game then it is the solution to the original game. It turns out that this is what IIA is all about.

5. Conclusion :- In this paper we provide two different characterizations of the IIA axiom which are intuitively appealing and geometrically reasonable. It is hoped that these two characterizations will help to put the IIA axiom in its proper perspective.

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