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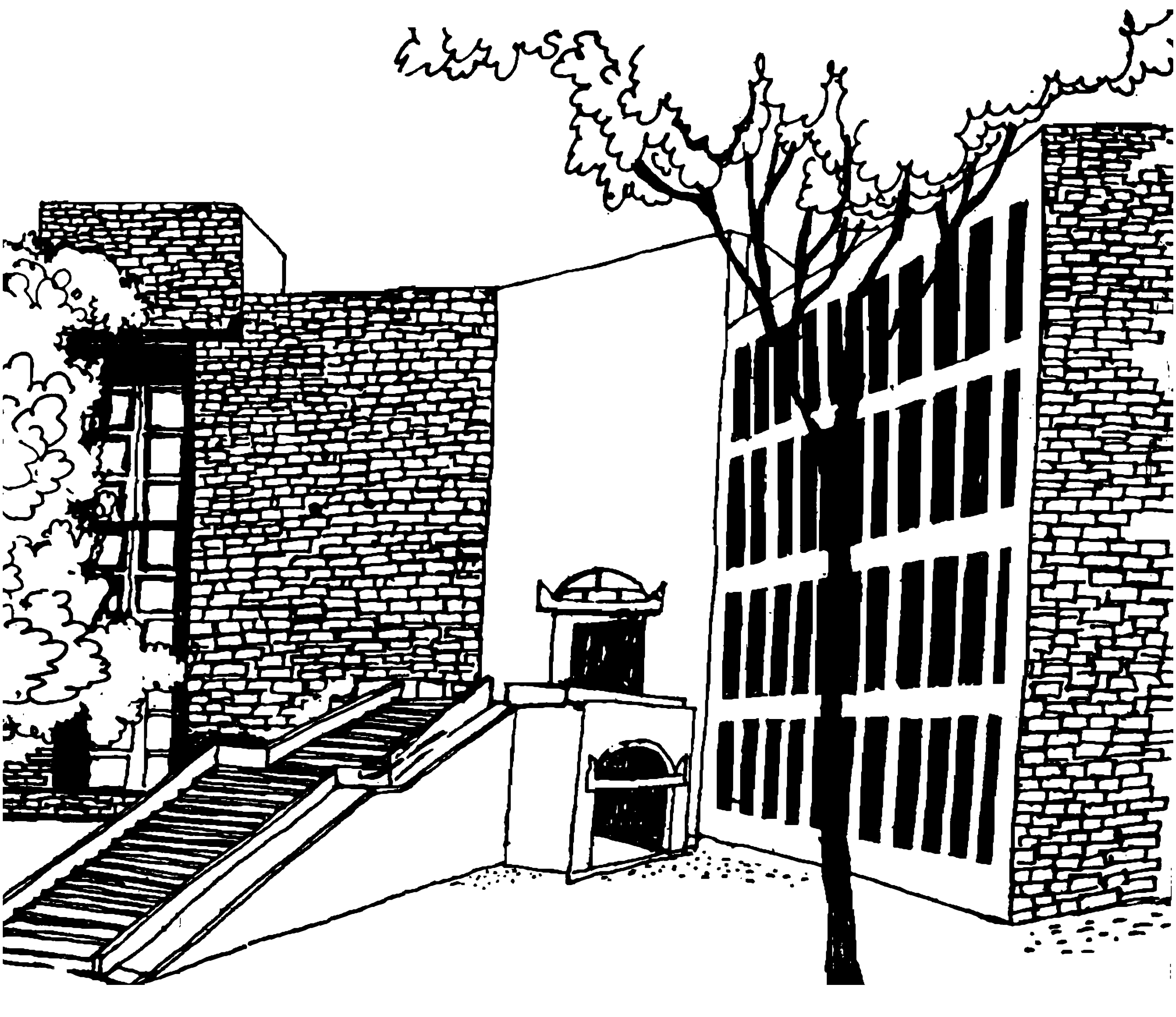

STRICTLY FAIR ALLOCATIONS IN ECONOMIES WITH A PUBLIC GOOD

By

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Abstract

In this paper we define the concept of strictly fair allocations in economies with public good and show that a equal income Lindahl Equilibrium allocation is strictly fair. Conversely if an allocation is strictly fair in every replication of the basic economy it must be an equal income Lindahl equilibrium allocation.

1. Introduction:

In this paper we investigate the economic equity and efficiency property of equal-income Lindahl equilibrium allocations in large economies. Following Zhou [1992] and Sato [1984] we adopt the concept of strict fairness to an economy consisting of two goods: one good being a pure private good (money) and the other a pure public good, the latter being produced from the former using a constant returns to scale technology.

Public goods form an undeniable part of social and economic reality and theories of distributive justice should naturally be extended to include public goods. Such attempts occur in Moulin [1987], Sato [1985, 1987], Otsuki [1992], Lahiri [1992a, 1992b, 1992c]. This paper is another attempt in the same direction.

2. The Model:

We shall draw on Sato [1987], Zhou [1992] and Moulin [1988] to postulate the following model:

The basic economy consists of a finite set of agents $N = \{1, \ldots, n\}$ and has two goods, one pure public good and one private good (money). The cost of producing y units of public good is cy units of money where $c > 0$. The aggregate endowment of money in the basic economy is $w > 0$.

An allocation for the basic economy is a vector $(x,y) \in \mathbf{R}_+^N \times \mathbf{R}_+$ where x_i is agent i's consumption of the private good and y is the social consumption of the public good in the economy. Given $w > 0$, the set of feasible allocations for the basic economy with aggregate endowment w is

$$F(w) = \{(x,y) \in \mathbf{R}_+^N \times \mathbf{R}_+ \;/\; \sum_{i \in N} x_i + cy = w\}$$

The consumption set of agent i in the basic economy is $\mathbf{R}_+^2$, and his preferences are characterized by a utility function u^i: $\mathbf{R}_+^2 \rightarrow \mathbf{R}$, where $u^i(x_i,y)$ gives agent i's satisfaction from consuming x_i units of money and y units of the public good.

We assume, each u^i: $\mathbf{R}_+^2 \rightarrow \mathbf{R}$ is

(i) twice continuously differentiable

(ii) quasi-concave

(iii) strictly monotone increasing

For each $i \in N$, given his consumption bundle $(x_i,y) \in \mathbf{R}_+^2$, let $\pi^i(x_i,y)$ be the marginal rate of substitution between the public good and the private good, i.e.

$$\pi^i(x_i,y) = \frac{\partial u^i(x_i,y)/\partial y}{\partial u^i(x_i,y)/\partial x}$$

Here, at the boundary of the domain, these partial derivatives are defined as the right-hand side ones. π^i is always strictly positive due to the strict monotonicity of the u^i's.

We are now in a position to define an <u>equal income Lindahl equilibrium</u> (EILE). A feasible allocation $(\bar{x},\bar{y}) \in F(w)$ is said to be an <u>equal income Lindahl equilibrium</u> if there exists a vector $(p_1,\ldots, p_n) \in \mathbf{R}_+^n \setminus \{0\}$ such that

(i) $(\bar{x}_i,y)$ maximizes $u^i(x_i,y)$ subject to $x_i + p_i\, y \leq \frac{w}{n}$, $x_i \geq 0$, $y \geq 0$ $\forall\, i = 1,\ldots,n$.

(ii) $\sum_{i=1}^{n} p_i = c$.

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(ii) $\sum_{i=1}^{n} p_i = c$.

We shall also require the concept of efficiency in the sequel: A feasible allocation $(\bar{x},\bar{y}) \in F(w)$ is said to be efficient if there is no other allocation $(x,y) \in F(w)$ such that $u^i(x_i,y) > u^i(\bar{x}_i,\bar{y})\ \forall\ i = 1, \ldots, n$.

An r-replica of the basic economy discussed above is an economy with r-agents each having the utility function u^i: $\mathbf{R}_+^2 \to \mathbf{R}\ \forall\ i \in N$ and aggregate initial endowment rw. The above definitions of an EILE and efficiency carry over to an r-replica economy for any positive integer 'r'.

We shall denote feasible allocations in an r-replica economy as follows:

$$(x^{(r)},y) \equiv ((x_i^j)_{j=1,\ldots,r}^{i=1,\ldots,n}, y) \in F(rw)$$

$$\Longleftrightarrow\ x_i^j \geq 0\ \forall\ (i,j),\ y \geq 0 \text{ and } \sum_{i=1}^{n} \sum_{j=1}^{r} x_i^j + cy \leq rw.$$

(here the subscript denotes a type of agent and the superscript the index of the agent within the type.)

The following definition is crucial to what follows.

Consider an r-replica of the basic economy defined above. An agent j of type i, denoted (i,j) envies a coalition S ((i,j) $\notin$ S) at any allocation $(x^{(r)},y)$ if

$$u^i(x_i^j,y) < u^i\left(\frac{1}{|S|}\sum_{(i',j')\in S} x_{i'}^{j'},\ \frac{1}{|S|}\frac{\sum_{(i',j')\in S}\pi^{i'}(x_{i'}^{j'},y)}{\pi^i(x_i^j,y)}\, y\right)$$

An allocation is strictly envy-free if no agent envies any other coalition.

Combining the concepts of efficiency and strict envy-freeness we arrive at the following definition:

Definition: An allocation $(x^{(r)},y) \in F(rw)$ is said to be strictly fair in the r-replica of the basic economy if it is both efficient and strictly envy-free in the r-replica of the basic economy.

3. The Main Results:

It is our purpose in this paper to establish the equivalence of the two concepts of an equal income Lindahl equilibrium and a strictly fair allocation. Sato [1987] establishes the equivalence of EILE with the concept of strong fairness. Lahiri [1992b] establishes equivalence of EILE with the concept of opportunity fairness. In a related endeavour Lahiri [1992c] establishes the equivalence of the concept of EILE and fairness for a continuum economy.

The following theorem is now immediate.

Theorem 1: Let (x,y) be an EILE allocation for the basic economy. Then $(x^{(r)},y)$ is a strictly fair allocation for the r-replica of the basic economy $\forall\, r \in N$, where $x_i^j = x_i\ \forall$ $j = 1, \ldots, r$, $i \in N$.

Proof: Suppose not. Observe that since (x,y) is an EILE allocation, if $(p_1, \ldots, p_n)$ is the associated vector of prices, our assumptions guarantee that

$$p_i = \frac{\partial u^i(x_i,y)/\partial y}{\partial u^i(x_i,y)/\partial x}\ \forall\, i = 1, \ldots, n.$$ Let $r \in N$ and $S \neq \phi$, $S \subseteq N \times \{1, \ldots,$ $r\}$ be such that $(i,j) \notin S$ and

$$u^i(x_i,y) < u^i\left(\frac{1}{|S|}\sum_{(i',j')\in S} x_{i'}^{j'},\ \frac{1}{|S|}\frac{\sum_{(i',j')\in S}\pi^{(i',j')}(x_{i'}^{j'},y)}{\pi^i(x_i,y)}\,y\right)$$

Since (x_0,y) is an EILE allocation,

$$x_i^0 + \pi^{i'}(x_{i'},y)\,y = \frac{w}{n} \quad \forall\,(i',j')\in S$$

$$\Rightarrow x_{i'} + \pi^i(x_i,y)\cdot\frac{\pi^{i'}(x_{i'},y)}{\pi^i(x_i,y)}\,y = \frac{w}{n} \quad \forall\,(i',j')\in S.$$

$$\Rightarrow \frac{1}{|S|}\sum_{(i',j')\in S} x_{i'}^{j'} + \pi^i(x_i,y)\cdot\frac{1}{|S|}\frac{\sum_{(i',j')}\pi^{i'}(x_{i'}^{j'},y)}{\pi^i(x_i,y)} = \frac{w}{n}$$

Since $p_i = \pi^i(x_i,y)$, the consumption bundle,

$$\left(\frac{1}{|S|}\sum_{(i',j')\in S} x_{i'}^{j'},\ \frac{1}{|S|}\frac{\sum_{(i',j')}\pi^{i'}(x_{i'}^{j'},y)}{\pi^i(x_i,y)}\right)$$ belongs to the budget set of (i,j) and

gives higher utility to (i,j) than (x_i^0,y), contradicting utility maximization by (i,j) at (x_i,y).

Since an EILE is efficient under our assumptions, we arrive at a contradiction. Hence $(x^{(r)},y)$ is strictly fair for all r.

Q.E.D.

The converse of this theorem establishes the required equivalence.

<u>Theorem 2</u>: Let $(x,y) \in F(w)$ with $y > 0$, and let $(x^{(r)},y)$ be as defined above. If $(x^{(r)},y)$ is a strictly fair allocation for the r-replica of the basic economy $\forall\, r \in N$, then (x,y) is an EILE allocation. (Here $x_i^j = x_i \;\forall\, (i,j) \in N \times \{1,\ldots,r\},\, r \in N$.)

<u>Proof</u>: Since $(x^{(r)},y)$ is strictly fair $\forall\, r \in N$, (x,y) is efficient in the basic economy. Let $p_i = \pi^i(x_i,y)$, $i \in N$. Since efficiency guaranties that (x_i,y) solves

$$u^i\,(x'_i,y') \rightarrow \max$$

$$\text{s.t.}\quad x'_i + p_i\, y' = x_i + p_i\, y$$

$$x'_i \geq 0,\; y' \geq 0.$$

$\forall\; i \in N$ (the second fundamental theorem), we need to show that

$$x_i + p_i\, y = \frac{w}{n} \quad \forall\, i \in N$$

(Since $(x,y) \in F(w),\; \sum_{i=1}^{n} x_i + y \sum_{i=1}^{N} p_i = w = \sum_{i=1}^{n} x_i + yc$ hence by $y > 0$, we have $\sum_{i=1}^{n} p_i = c$).

Suppose $x_i + p_i\, y < \frac{w}{n}$ for some $i \in N$. Thus there exists $i' \in N$, such that

$$x_{i'} + p_{i'}\, y > \frac{w}{n}$$

$$\therefore\; x_{i'} + p_{i'}\, y > x_i + p_i\, y \rightarrow x_{i'} + p_i \left(\frac{p_{i'} y}{p_i}\right) > x_i + p_i \left(\frac{p_i y}{p_i}\right)$$

The function $(\alpha,\beta) \mapsto u^i\left(\alpha,\frac{\beta}{p_i}\right) :: \mathbb{R}^2_+ \to \mathbb{R}$ is differentiable strictly increasing

and quasi-concave. Since $x_{i'} + p_{i'}\, y > x_i + p_i\, y$, $\exists\, t \in (0,1)$ such that

$$u^i\left(tx_{i'} + (1-t)x_i\ ,\ \frac{(tp_{i'} + (1-t)p_i)y}{p_i}\right) > u^i(x_i,y).$$

By continuity there exists a rational no: $\frac{s}{r} \in (0,1)$, such that

$$u^i\left(\frac{s}{r}x_{i'} + \left(1-\frac{t}{r}\right)x_i\ ,\ \frac{(sp_{i'} + (r-s)p_i)y}{rp_i}\right) > u^i(x_i,y). \quad \text{Here } r,\ s \in \mathbf{N}.$$

Consider an r-replica of the basic economy and let S consist of s agents of type i′ and (r-s) agents of type i. Since r > s, there is a type i agent who does not belong to S and who envies coalition S at $(x^{(r)},y)$, contradicting that $(x^{(r)},y)$ is strictly fair $\forall\, r \in \mathbf{N}$.

Q.E.D.

4. Conclusion:

In this paper we establish the equivalence of the EILE concept with the concept of strict fairness in an economy with a public good. The equivalenee theorem has the flavour of the Debreu-Scarf limit theorem for the core of a replica economy. The technique of proof is also similar: we simply consider a large enough replica of the basic economy to violate strict fairness in case we consider a non EILE allocation. The implication of this result in the theory of distributive justice for a mixed economy is that the equal income Lindahl equilibrium mechanism qualifies not only as an efficient resource allocation mechanism, but also as the only mechanism with a desirable "fairness" property.

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