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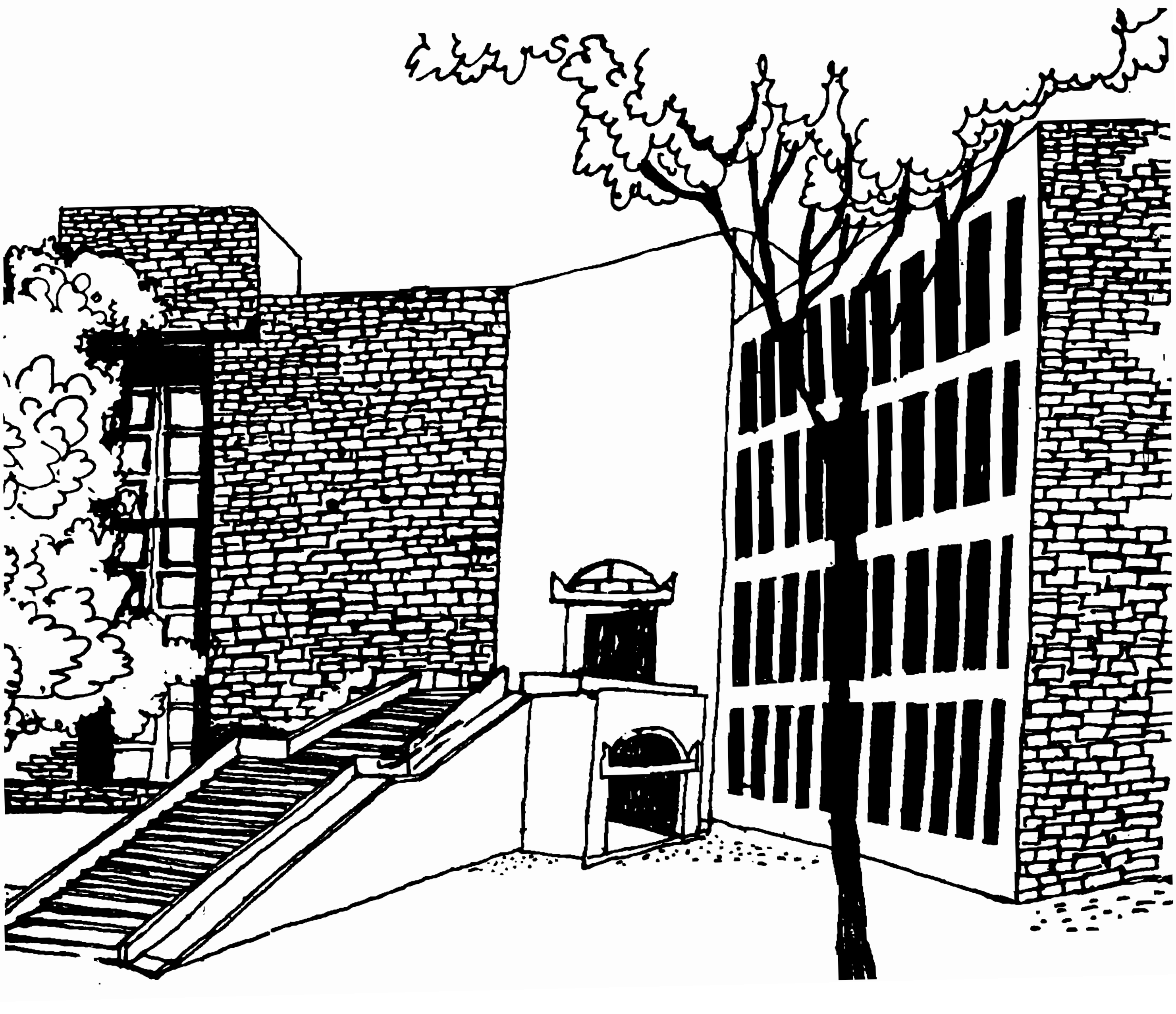

EXISTENCE OF EQUILIBRATED STATES IN MULTI-CRITERIA DECISION MAKING PROBLEMS

By

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My interest in multi-criteria decision making is mainly due to my interactions with my colleague Professor P.R. Shukla. I have had the benefit of many useful discussions on this topic with Professor Prakash Abad, as well. The application of multi-criteria decision making studied in this paper, is in line with my current research interests, for which I would like to thank amongst others Professor J. Jordan.

Abstract

In this paper we propose a general model of multi-criteria decision making and prove that an equilibrated state exists under natural assumptions. As an application of this theory, we prove the existence of budget constrained Pareto efficient allocations using methods developed in the paper.

1. Introduction:

In a recent paper, Polterovich [1990], proves the existence of equilibrated states for group decision making problems. In that paper, each agent was being evaluated by an external expert, through one criteria. The natural question that now arises is if a similar theory with more than one criteria for each agent, admits of a similar existence result.

In this paper, we propose a theory of group decision making with more than one criteria for each agent, and show that under natural assumptions an equilibrated state does exist for each such problem. We close the paper with an application and some concluding remarks.

2. The Model:

As in Polterovich [1990], we assume that $Z(\subseteq \mathbf{R}^l)$ is a set of feasible states. Let I be a finite index set of agents and for each $k \in I$, U_k: $Z \to \mathbf{R}$ be a continuous utility function for agent $k \in I$, which represents his preferences over alternative feasible states.

A state $z \in Z$ is called Pareto optimal if there does not exist $z' \in Z$ with $U_k(z') \geq U_k(z)$ $\forall k \in I$ with at least one strict inequality. We say that a state $z \in Z$ is weakly Pareto optimal if there does not exist $z' \in Z$ with $U_k(z') > U_k(z)$ $\forall k \in I$.

We assume that Z is a convex subset of $\mathbf{R}^l$.

A function f: $Z \to \mathbf{R}$ is said to be strictly quasi-concave if $\forall z, z' \in Z$ with $z \neq z'$, $f(tz + (1-t)z') > \min.\{f(z), f(z')\}$ $\forall t \in (0,1)$.

A function f: $Z \to \mathbf{R}$ is said to be semi-strictly quasi-concave if $\forall z, z' \in Z$ with $f(z) \neq f(z')$, $f(tz + (1-t)z') > \min.\{f(z), f(z')\}$ $\forall t \in (0,1)$.

Let $\Delta = \left\{p \in \mathbf{R}_+^{\#I} \;/\; \sum_{i \in I} p_i = 1\right\}$. The following proposition is significant to what follows:

Proposition 1:

(a) If z^* solves

$$\sum_{k \in I} p_k U_k(z) \rightarrow \max$$
$$\text{s.t. } z \in Z$$

for some $p = (p_k)_{k \in I} \in \Delta$, then z^* is weakly Pareto optimal. If $p_k > 0 \; \forall \; k \in I$, then z^* is Pareto optimal.

(b) If U_k: $Z \rightarrow \mathbf{R}$ is strictly quasi-concave $\forall \; k \in I$ and z^* solves

$$\sum_{k \in I} p_k U_k(z) \rightarrow \max$$
$$\text{s.t. } z \in Z$$

for some $p = (p_k)_{k \in I} \in \Delta$, then z^* is Pareto optimal.

Proof:

(a) The proof is obvious.

(b) Suppose z^* solves the problem in (b) and suppose $\exists \; z' \in Z$ such that $z' \neq z^*$ and $u_k(z') \geq u_k(z^*) \; \forall \; k \in I$. Then for $t \in (0,1)$, $tz' + (1-t)z^* \in Z$ and $u_k(tz' + (1-t)z^*) > u_k(z^*) \; \forall \; k \in I$, contradicting that z^* solves the above problem. Hence z^* must be Pareto optimal and in fact a unique solution to the problem in (b).

Q.E.D.

For each $k \in I$, let g_k: $Z \rightarrow \mathbf{R}^r$ be a continuous criteria function for agent k. Thus $g \equiv (g_k)_{k \in I}$: $Z \rightarrow (\mathbf{R}^r)^{\#I}$ is a criteria function for the group with $r \geq 1$.

Definition: A state $z \in Z$ is said to be equilibrated if (a) z is Pareto optimal; (b) $g_k(z) \leq 0 \; \forall \; k \in I$.

3. Existence of Equilibrated States:

In order to prove our basic existence theorem we make the following simplifying assumptions:

A.1: $\forall\, k \in I$, U_k: $Z \to \mathbb{R}$ is strictly quasi-concave.

A.2: Z is a compact, convex subset of $\mathbb{R}^l$.

A.3: $\sum_{k \in I} \max_j \left\{ g_k^j(z) \right\} \leq 0 \ \forall\, z \in Z$ which are Pareto optimal.

A.4: If z maximizes $\sum_{k' \in I \setminus \{k\}} p_{k'} U_{k'}(z')$ subject to $z' \in Z$ where $\sum_{k' \in I \setminus \{k\}} p_{k'} = 1$, then $g_k^j(z) > 0$ for some $j \in \{1, \ldots, r\}$.

We now prove our main theorem:

Theorem 1: Under assumptions A.1, A.2, A.3 and A.4, there exists an equilibrated state.

Proof: For each $p \in \Delta$, let $z(p)$ be the unique solution to

$$\sum_{k \in I} p_k U_k(z) \to \max$$

$$\text{s.t. } z \in Z$$

The function z: $\Delta \to Z$ thus defined is continuous. Consider the function f: $\Delta \to \Delta$,

$$f_k(p) = \frac{p_k + \max\left(g_k^1(z(p)), \ldots, g_k^r(z(p)), 0\right)}{1 + \sum_{k' \in I} \max\left(g_{k'}^1(z(p)), \ldots, g_{k'}^r(z(p)), 0\right)}, \quad k \in I$$

and $f = (f_k)_{k \in I}$. Clearly f is a continuous function from Δ to Δ. Thus, by Brouwer's fixed point theorem, there exists $p^* \in \Delta$ such that $f(p^*) = p^*$.

Case 1: $p_k^* = 0$. Then

$$\max\left(g_k^1(z(p*)), \ldots, g_k^r(z(p*)), 0\right) = 0$$

$$\text{i.e. } g_k(z(p*)) \le 0.$$

But if $p_k^* = 0$, then $z(p^*)$ solves

$$\sum_{k' \in I \setminus \{k\}} p_{k'}^* U_{k'}(z) \rightarrow \max$$

$$\text{subject to } z \in Z.$$

Further $\sum_{k' \in I \setminus \{k\}} p_{k'}^* = 1.$

Thus by A.4, there exists $j \in \{1, \ldots, r\}$ such that $g_k^j(z(p^*)) > 0$, which is a contradiction. Hence $p_k^* = 0$ is inadmissable.

Case 2: $p_k^* = 1$. Then $p_{k'}^* = 0 \;\forall\; k' \in I \;\; \{k\}$ which is inadmissable. Hence $p_k^* = 1$ is also inadmissable.

Case 3: $0 < p_k^* < 1 \;\forall\; k \in I$.

$$\therefore p_k^* \sum_{k' \in I} \max\left\{g_{k'}^1(z(p*)), \ldots, g_{k'}^r(z(p*)), 0\right\}$$

$$= \max\left\{g_{k'}^1(z(p*)), \ldots, g_k^r(z(p*)), 0\right\} \;\forall\; k \in I.$$

$$\therefore \max_j \left\{g_k^j(z(p*))\right\} \sum_{k' \in I} \max\left\{g_{k'}^1(z(p*)), \ldots, g_{k'}^r(z(p*)), 0\right\}$$

$$= \max_j \frac{\left\{g_k^j(j(p*))\right\}}{p_k^*} \max\left\{g_k^1(z(p*)), \ldots, g_k^r(z(p*)), 0\right\}.$$

Summing over 'k' we get

$$0 \geq \sum_{k \in I} \max_{j} \left\{ g_k^j(z(p*)) \right\} \max \left\{ g_k^1(z(p*)), \ldots, g_k^r(z(p*)), 0 \right\} \ \forall\, j \in \{1, \ldots, r\}$$

i.e. $g_k^j(z(p*)) \leq 0 \ \forall\, k \in I, \ \forall\, j \in \{1, \ldots, r\}$.

This proves the theorem.

Q.E.D.

If we relax our assumption on preferences and modify it to read as:

A.5: $\forall\, k \in I$, U_k: $Z \to \mathbf{R}$ is semi-strictly quasi-concave;

Theorem 2: Under assumptions A.2, A.3, A.4 and A.5, there exists an equilibrated state.

Proof: To prove Theorem 2, we make use of the following theorem due to Cellina [1969]:

Theorem: Let K be a compact set (say in some Euclidean space), T a compact convex set of $\mathbf{R}^l$, and Q: $K \twoheadrightarrow T$ be a nonempty valued, convex valued correspondence such that the graph $\Gamma_Q = \{(x,y) \in K \times T \,/\, y \in Q(x)\}$ is compact. Then given $\epsilon > 0$ there is a continuous function h^ϵ: $K \to T$ such that $\Gamma_{h^\epsilon} \subseteq B_{2\epsilon}(\Gamma_Q)$ where $B_{2\epsilon}(\Gamma_Q)$ is the open set of all points in K x T within 2ϵ of Γ_Q.

For $p \in \Delta$, then $z(p) = \left\{ z \in Z \,/\, \sum_{k \in I} U_k(z) \geq \sum_{k \in I} U_k(z') \ \forall\, z' \in Z \right\}$. The correspondence z: $\Delta \twoheadrightarrow Z$ thus defined is nonempty valued (since each U_k, $k \in I$ is continuous and Z is compact), convex valued (since each U_k, $k \in I$ is semi-strictly quasi-concave and Z is convex) and has a closed graph in Z. Thus by Cellina's theorem, $\forall\, \epsilon > 0$, there exists a continuous function z^ϵ: $\Delta \to Z$ such that $\Gamma_{z^\epsilon} \subseteq B_{2\epsilon}(\Gamma_z)$. Define f^ϵ: $\Delta \to \Delta$ as follows:

$f^{\epsilon} \equiv \left(f_k^{\epsilon}\right)_{k \in I}$, where

$$f_k^{\epsilon}(p) = \frac{p_k + \max\left(g_k^1(z^{\epsilon}(p)), \ldots, g_k^r(z^{\epsilon}(p)), 0\right)}{1 + \sum_{k' \in I} \max\left(g_{k'}^1(z^{\epsilon}(p)), \ldots, g_{k'}^r(z^{\epsilon}(p)), 0\right)}, \quad k \in I$$

f^{ϵ} is continuous. Hence by Brouwer's fixed point theorem, there exists $p^{\epsilon} \in \Delta$ such that $f^{\epsilon}(p^{\epsilon}) = p^{\epsilon}$. Let $\{\epsilon_j\}_{j \in N}$ be a decreasing sequence of positive reals such that $\lim_{j \to \infty} \epsilon_j = 0$. Since $\{p^{\epsilon_j}\}_{j \in N} \subseteq \Delta$, and Δ is compact, there exists a subsequence of $\{p^{\epsilon_j}\}_{j \in N}$ converging to a point $p^* \in \Delta$. Since $\{z^{\epsilon_j}\}$ converges uniformly to $z: \Delta \twoheadrightarrow Z$ and $f^{\epsilon_j}(p^{\epsilon_j}) = p^{\epsilon_j} \; \forall \, j$, we must have

$$p^* \in \left\{ \left(\frac{p_k^* + \max\left(g_k^1(z), \ldots, g_k^r(z), 0\right)}{1 + \sum_{k' \in I} \max\left(g_{k'}^1(z), \ldots, g_{k'}^r(z), 0\right)} \right)_{k \in I} \Big/ \, z \in z(p^*) \right\}$$

From here on the proof runs exactly as the proof of Theorem 1, to establish the existence of an equilibrated point (by Proposition 1(a)), since $p_k^* > 0 \; \forall \in I$.

Q.E.D.

4. <u>An Application: Existence of a Budget Constrained Pareto Efficient Allocation</u>:

In this section we apply Theorem 2, to show that a budget constrained Pareto efficient allocation in the sense of Balasko [1979], Keiding [1981], Svensson [1984] exists in the framework of a distribution economy as studied by Malinvaud [1972], Lahiri [1993].

We consider an economy consisting of a finite number of consumers, indexed by members of a set I. The consumption set of each consumer $i \in I$, is the non-negative orthant of

Euclidean, n-space: $\mathbf{R}^n_+$. Each consumer $i \in I$, has a fixed positive initial endowment of income $w_i > 0$. The aggregate initial endowment of resources in the economy is $\omega \in \mathbf{R}^n_{++} = \{x \in \mathbf{R}^n_+ / x_j > 0 \ \forall \ j \in \{1, \ldots, n\}\}$.

The preferences of each consumer $i \in I$ over vectors of consumption goods are defined by a utility function u_i: $\mathbf{R}^n_+ \to \mathbf{R}$. We assume that for each $i \in I$, u_i is continuous, increasing (i.e. $x, y \in \mathbf{R}^n_+$, $x \gg y \Rightarrow u_i(x) > u_i(y)$) and semi-strictly quasi-concave.

An allocation is a vector $x = (x^i) \in (\mathbf{R}^n_+)^{\#I}$. An allocation x is said to be feasible if $\sum_{i \in I} x^i = \omega$.

Let $\hat{p} \in \mathbf{R}^n_{++}$ be a vector of strictly positive commodity prices, such that $\hat{p} \cdot \omega = \sum_{i \in I} w_i$.

Definition: An allocation $(\hat{x}) \in (\mathbf{R}^l_+)^{\#I}$ is said to be a budget constrained Pareto efficient allocation (BCPE allocation) if

(i) $\hat{x}$ is feasible and Pareto optimal.

(ii) $\hat{p} \cdot \hat{x}^i = w_i \ \forall \ i \in I$.

We now prove that a BCPE allocation exists.

Theorem 3: A BCPE allocation exists for the distribution economy defined above.

Proof: Let Z be the set of all feasible allocations and for each $i \in I$ define g_i: $Z \to \mathbf{R}$ as follows:

$$g\left((x^i)_{i \in I}\right) = w_i - \hat{p} \cdot x^i \ , \ (x^i)_{i \in I} \in Z.$$

It is easy to verify that g is continuous and verifies A.3 and A.4, Z is compact-convex (and hence verifies A.2) and the utility function of each agent satisfies A.5. Thus by Theorem

2, a weakly equilibrated state exists. Let this state be $(\hat{x}^i)_{i\epsilon I} \epsilon$ Z. Further $w_i - \hat{p}\cdot\hat{x}^i \leq 0 \; \forall \; i \; \epsilon$ I, implies $\hat{p}\cdot\hat{x}^i = w_i$. Hence $(\hat{x}^i)_{i\epsilon I}$ is a BCPE allocation.

Q.E.D.

5. Conclusion:

We have thus managed, to extend Polterovich's definition of an equilibrated state to truly "multi-criteria optimization problems". These are multi-criteria problems, not simply because there is more than one agent with a utility function assigned to each, but because there are one or more external criteria which an equilibrated state needs to satisfy for each agent. As the application in section 4 shows, assumption A.4 is really a mild-regularity assumption saying that if an agent is ignored then the resulting state cannot be an equilibrated state.

We have also chosen to avoid the needless technical complexity of having criteria correspondences. The reasons are twofold: since we consider a vector of criteria we are admitting indirectly more than one, but a finite number of indices which the expert has to process for each agent; most applications that Polterovich [1990] as well as we do consider, can be dealt with criteria functions. However, our method though different from the one cited above, can be easily modified to accommodate criteria correspondences.

Appendix

In this section we provide an alternative (equivalent) proof of Theorem 2, using Kakutani's fixed point theorem.

Theorem A :- Under assumptions A.2, A.3, A.4 and A.5, there exists an equilibrated state.

Proof :- Define the correspondence $z: \Delta \to\to Z$ as in the proof of Theorem 2, in section 3. Now define the correspondence $f: \Delta \times Z \to\to \Delta \times Z$ as follows :

$$f(p,z) = \left(\frac{p_k + \max(g^1_k(z), \ldots, g^r_k(z), 0)}{1 + \Sigma_{k' \in I} \max(g^1_{k'}(z), \ldots, g^r_{k'}(z), 0)} \right)_{k \in I} \times z(p)$$

f is well defined, non-empty valued, convex-valued and has a closed graph. Hence by Kakutani's fixed point there exists $(p^*, z^*) \in \Delta \times Z$, such that $(p^*, z^*) \in f(p^*, z^*)$. From here on the proof proceeds as in the proof of Theroem 1, once it is observed that $z^* \in z(p^*)$, to establish the existence of an equilibrated state.

Q.E.D.

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