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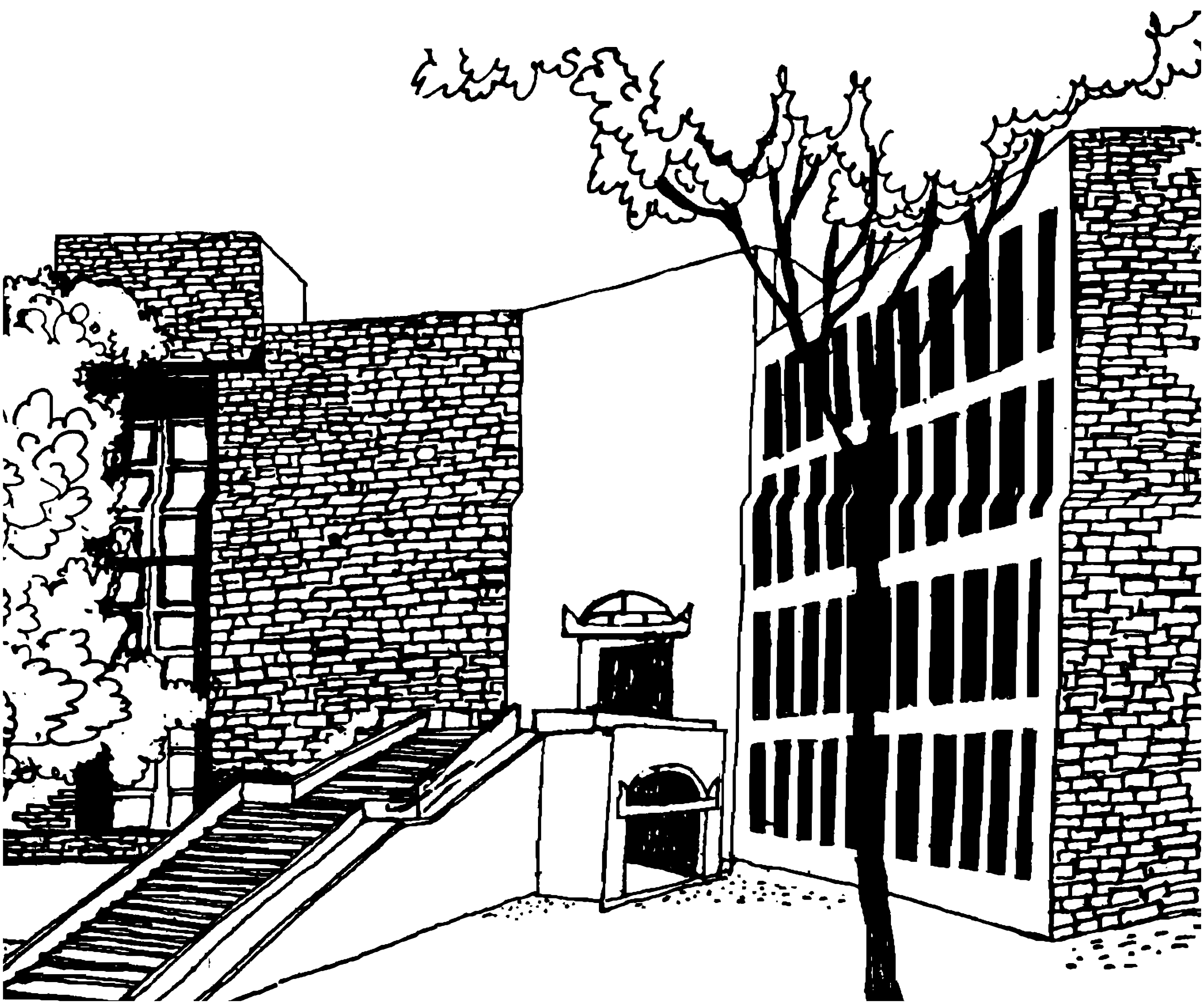

RESTRICTED EXPANSION INDEPENDENCE FOR CHOICE PROBLEMS

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ABSTRACT

In this paper we propose restricted expansion independence as a criterion which may be satisfied by desirable choice functions and axiomatically characterize the proportional solution by using this criterion. We also show that the proportional solution satisfies an improvement sensitivity property on a reasonable domain. The theory of solutions to choice problems is used in the paper to define solutions for coalitional bargaining problems.

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1. INTRODUCTION :

In recent years, considerable work has been done in axiomatically characterising solutions to choice problems in Euclidean spaces. Briefly, a choice problem is an ordered pair of a set of alternatives in a finite dimensional Euclidean space (from which a choice has to be made) and a vector in the same space (variously referred to as a claims point, target point) which affects the choice. It is difficult to pinpoint the exact genesis of the analytical study of choice problems. However, Chun and Peters (1988), Bossert (1992), Bossert (1993) are some early studies in this direction, followed by Lahiri (1994 a,b) where also some recent axiomatic characterizations of solutions to choice problems are available. in an earlier paper, Lahiri (1991), an interesting monotonicity property of solutions to such problems proposed in a related context by Yu (1973) finds expression. The line of approach adopted in all but the last work has its roots in the study of bargaining problems, which by now has been surveyed exhaustively by several authors.

In Lahiri (1994a) we provide an axiomatic charaterization of what is known in the literature as the proportional solution. This solution associates with each choice problem, the point on the efficient frontier of the feasible set which lies on the line joining the origin to the target point. This solution closely generalizes the relative egalitarain solution due to Kalai and →

——————→ Smorodinsky (1975), as do so many others including the solution proposed in Lahiri (1994 b). A crucial assumption used in this axiomatic characterization is restricted monotonicity, i.e. if two choice problems have the same target point, and one feasible set is contained in the other, then no co-ordinate of the solution to the smaller problem can exceed the coresponding co-ordinate of the solution to the larger problem.

We have nothing whatsoever against the above axiom. However, it would be desirable to find out if a weaker axiomatization could do the job. Such an axiomatization is basically what is proposed in this paper. Some what modifying a similar axiom for bargaining problems suggested in Lahiri (1995) we have an axiom called restricted expansion independence which now replaces restricted monotonicity. It is instructive to note that restricted monotonicity along with efficiency (a property common to both axiomatizations) implies restricted expansion independence, so that now we have a proof which works for either axiomatic characterization. The proof itself is an adaptation of a proof which appears in Moulin (1988), which was earlier used in characterizing the egalitarian solution. The interesting thing to note here is that although the proportional solution generalizes the relative egalitarian solution, identical techniques do not hold for the axiomatic characterization of the latter.

It has been argued in several places, that axiomatic choice theory is a theory of concensus and breaks down as soon as one admits coalitions to form as in coalitional bargaining games. Hence independent axiomatic characterization of solution to coalitional barganing games have been provided, where the solution to the game is only remotely (if at all) connected with any underlying choice problems.

Our stand point in this paper is somewhat different. We feel that when a coalition has to decide on a solution, it would treat the problem as one of solving a choice problem. In effect the coalition should view itself as a given entity and whatever coalition formation it should guard against, should be appropriately summarized in a target payoff for each members of the coalition. Assuming this position, we provide a method of obtaining a solution to a coalitional bargaining game. In order to exhibit the desirability of using the proportional solution for the purpose of obtaining a value allocation for a coalitional bargaining problem by our iterative method, we show that it satisfies an improvement sensitivity propertey in the final section of the paper.

2. THE FRAME WORK :

Let M, a finite subset of the natural numbers denote an index set. Let $R^M{+}$ be the set of all functions from M to R+ (the set of non-negative reals). If $x \in R^M{+}$ and i $\in$

M, then $x(i)$ is written simply as x_i. A choice problem for M is an ordered pair (S,c) such that

(i) $c \in R^M_+$ with $c_i > 0 \; \forall \, i \in M$

(ii) S is a non-empty, compact, convex subset of $\mathbb{R}^M_+$

(iii) S is strictly comprehensive, i.e. (a) $y \in S$, wherever $y \in R^M_+$ and $y \leq X$ for some $x \in S$, (b) $x, y \in S$, $x \leq y$, x is not equal to y ====> $z \in S$ with $Z_i > x_i \; \forall \, i \in M$.

Let $\mathcal{L}$ (M) denote the set of all choice problem for M.

A solution on $\mathcal{L}$ (M) is any function F : $\mathcal{L}$ (M) --> R^M_+ such that $F(S,c) \in S \; \forall \, (S,c) \in \mathcal{L}(M)$

The proportional solution $P : \mathcal{L}(M) \rightarrow R^M_+$ is defined as follows :

$P(S,c) = t(S,c)\, c$

where $t(S,c) = \max \{t \in R_+ \,/\, tc \in S\}$ and $(S,c \in \mathcal{L}(M)$.

An axiomatic characterization of the proportional solution is available in Lahari (1994). It is easy to see that the proportional solution satisfies the following axioms :

Axiom 1 (Efficiency) : $\forall (S,c) \in \mathcal{L}(M)$, $y \in S$, $y \geq F(S,c)$ ====> $F(S,c) = y$

Axiom 2 (Scale invariance) : $\forall \, (S, c) \in \mathcal{L}(M)$, $a \in R^M_{++}$, $F(aS, ac) = a\,F(S,c)$

(Here for $x \in R^M_+$, $a \in R^M_{++}$, $(ax)_i = a_i x_i$, $\forall \, i \in M$; for $S \in R^M_+$, $aS = \{ax \,/\, x \in s\}$)

Axiom 3 (Anonymity) : $(S,c) \in \mathcal{L}(M)$ $\forall i,j \in M$, if (T,d) arises from (S,c) by interchanging the ith and j th co ordinates of the points of S and of c, we have $F_i(S,c) = F_j(T,d)$, $F_j(S,c) = F_i(T,d)$ and $F_k(S,c) = F_k(T,d)$ $\forall k \neq i,j$.

Axiom 4 (Restricted Expansion Independence) If (S,c), $(T,c) \in \mathcal{L}(M)$, $S \subseteq T$ and $y \in T$, $y \geq F(S,c)$ implies $y = F(S,c)$ then $F(S,c) = F(T,c)$.

3. A characterization of the Proportional Solution :

Theorem 1 : The only solution on $\mathcal{L}(M)$ to satisfy Axioms 1 to 4 is P.

Proof : Let $(S,c) \in \mathcal{L}(M)$. If $S = \{o\}$, there is nothing to prove. Hence assume $S \neq \{o\}$. Thus there exists a strictly positive vector in S. Hence $P_i(S,c) > 0$, $\forall i \in M$. By Axiom 2 assume $c_i = c_j \forall i,j \in M$. Thus $P_i(S,c) = P_j(S,c) = \mu > o$ $\forall i,j \in M$. Let $e \in R^{M}_{+}$ be such that $e_i = 1$ $\forall i \in M$. Clearly $\forall i \in M$, there exists a vector $a^i \in S$ with $a_i^i < \mu$ and $a_j^i > \mu$ if $j \neq i$. Let $\alpha = \min_{i \neq j} a_j^i$. Clearly $\alpha > \mu$. Let $V^i \in R^M_+$ with $V_j^i = \alpha$ if $j \neq i$, $v_i^i = 0$. Then $V^i \leq a^i$ and by comprehensiveness of S, $V^i \in S$, $\forall i \in M$.

Let T = Convex hull $\{0, [(v^i), i \in M], \mu e\}$. Thus $T \subseteq S$. By Axiom 3, $F_i(T,c) = F_j(T,c)$ $\forall i,j \in M$, where F is any solution satisfying Axioms 1 to 4. By Axiom 1, $F(T,c) = \mu e = P(S,c)$. Since P satisfies Axiomn 1, $y \in S$, $y \geq F(T,c)$ $\Longrightarrow y = F(T,c)$. By Axiom 4, $F(S,c) = F(T,c) = P(S,c)$.

4. A Theory of Value For Coalitional Bargaining Games : The solution method discussed above leads to a theory of value for coalitional bargaining games. Let us first define a coalitional bargaining game.

Let N be a finite set of players. N is called the player set. A coalitional bargaining game is a set valued function V defined on all non-empty subsets of N such that $\forall$ M $\subseteq$ N, M $\neq$ $\emptyset$, V(M) $\subseteq$ R^{M}_{+}. (A non- empty subset of N is called a coalition). We shall consider coalitional barganing games V which satisfy the following property : (a) V ({i}) = {o} $\forall$ i $\in$ N.

(b) $\forall$ M $\subseteq$ N, M $\neq$ $\emptyset$, $|M| \geq 2$, V(M) is compact, convex, strictly comprehensive and V(M) $\neq$ {O}, Let $\mathcal{G}$ denote the above set of coalitional bargaining games (here after referred to simply as games).

A domain is any subset D of $\mathcal{G}$

A solution on D is any function G : D ---> R^{N}_{+} such that G (V) $\in$ V(N) $\forall$ V $\in$ D.

We are interested basically in finding a method of implementing a solution on D by a collection of solutions to choice problems. We thus define a method $\mathfrak{F} = \{F_M\}$

$$M \subseteq N, \quad M \neq \emptyset, \quad |M| \geq 2$$

as a collection of solution to choice problems where each F_M is a solution on $\mathcal{C}$ (M), satisfying F_M (T) >> 0 T $\in$ $\mathcal{C}$ (M), T $\neq$ {O}.

For $M \subseteq N$, $M \neq \emptyset$, $|M| = 2$, let $c^M = F_{M}(V(M), u(V(M)))$,

where

$u(V(M)) = (u^M_i)$ with $u^M_i = \max_{x \in S} x_i \;\; \forall i \in M$. $u(V(M))$ is called the utopia point of $V(M)$.

$$\text{Let } c^2_i = \frac{1}{(n-1)} \sum_{M \ni i} c_i^M$$

For $M \subseteq N$, $M \neq \emptyset$, $|N| > |M| \geq 2$, let $c^M = F_M(V(M), c_M^{|M|-1})$,

$$\text{and } \quad c_i^{|M|} = \frac{1}{\binom{|N|-1}{|M|-1}} \sum_{M \ni i} c_i^M$$

The method $\mathcal{F}$ implements the solution G on D if

$G(V) = F_N(V(N), c_N^{|N|-1}) \quad \forall V \in D.$

Alternatively, the method $\mathcal{F}$ defines the solution G on D if

$G(V) = F_N(V(N), c_N^{|N|-1}) \quad \forall V \in D.$

In the above we have in the process of defining the vectors $c_M^{|M|-1}$, $M \subseteq N$, $M \neq \emptyset$, assumed that each agent i can with equal probability associate with any coalition of the same coardinality and $c^{|M|-1}$ is the expected payoff to player i, if he is to join a coalition of size $|M|-1$. When the payoffs for an agent in a coalition is determined, it is reasonable to assume (a) that the formation of the coalition is an accepted fact; and (b) expected payoffs from sub coalitions of smaller size affect the payoffs to the agents belonging to the coalition. A solution to a choice problem would then assign a "fair" outcome to the coalition, based on the above considerations.

Example :- Let $N = \{1,2,3\}$; Let $V(\{i\}) = \{0\}$ $\forall\, i \in N$,

$$V(\{i,j\}) = \{ (x_i, x_j) \in R_+^{\{i,j\}} \,/\, x_i + x_j \leq v(\{i,j\}) \} \quad \forall\, i,j \in N$$

$$V(\{1,2,3\}) = \{ x \in R_+^{\{1,2,3\}} \,/\, x_1 + x_2 + x_3 \leq v(\{1,2,3\}) \}$$

Let F_M be the proportional solution $\forall\, M \subseteq N$, $M \neq \emptyset$ $|M| \leq 2$.

$$e_i^{\{i,j\}} = \frac{v(\{i,j\})}{2} \qquad \forall\, i,j \in N$$

$$e^2_i = \frac{v(\{i,j\})}{4} + \frac{v(\{i,k\})}{4} \quad \text{for } i,j,k \in N,\ j \neq k,\ j \neq i,\ k \neq i.$$

$$e_i^3 = \frac{e^2_i}{e^2_1 + e^2_2 + e^2_3} \; v(\{1,2,3\}) \qquad \forall\, i \in N.$$

Now, $e^2_1 + e^2_2 + e^2_3 = \frac{1}{2} [v(\{i,j\}) + v(\{i,k\}) + v(\{j,k\})]$

$$\text{Hence } e_i^3 = \frac{1}{2} \frac{v(\{i,j\}) + v(\{i,k\})}{v(\{i,j\}) + v(\{i,k\}) + v(\{j,k\})} \, v(\{1,2,3\}) \geq 0 \qquad \forall\, i \in N$$

Clearly $\Sigma^3_{i=1} e^3_i = v(\{1,2,3\})$

Now suppose $v(\{i,j\}) = b$ $\forall\, i,j \in \{1,2,3\}$.

Therefore $e^3_i = 1/3 \; v(\{1,2,3\})$

If $\mathcal{V}(\{1,2,3\}) \geq 3/2\, b$, then $(\mathcal{C}^3_1, \mathcal{C}^3_2, \mathcal{C}^3_3)$ belongs to the core of the game V.

It is pertinent to mention at this stage, that the aggregation **procedure** used for finding the points, $\mathcal{C}^t_i$, $i \in N$, $t = 2, \ldots, |N| - 1$ can be considerably generalized. Let $C_i(t)$ be the coalitions of size t containing i. Let $X_i(t) = \prod_{M \in C_i(t)} R^M_{++}$. We may define an aggregation rule for i at stage t, to be a function f^t_i : $X_i(t) \dashrightarrow R_{++}$ sch that $c \in X_i(t)$, $d \in X_i(t)$, $c^M_i = d^M_i\ \forall M \in C_i(t)$ implies $f^t_i(c) = f^t_i(d)$ i.e. the aggregation rule for i at any stage is independent of what others in any coalition Jcontaining i are obtaining. This is really not a very serious restriction. Yet another example of such a rule could be $f^t_i(c) = \max\{\mathcal{C}^M_i \,/\, M \in Ci(t)\}$, $\forall i \in N$. This assumption sharpens the underlying theory considerably. We have however, chosen a specific aggregation rule at each stage in order to focus on the solution concept rather than on the aggregation procedure.

5. Improvement Sensitivity Property of the Proportional Solution :

In this section we extend the definition of impovement sensitivity for bargaining problems as stated for instance in Peters [1992] to exhibit the same property on a Subdomain of $\mathcal{B}(M)$.

Let $D = \{(S,c) \in \mathcal{L}(M) \,/\, c \in R^M_+ \setminus S\}$. D is the set of choice problems in $\mathcal{L}(M)$ for which the target point is not attainable.

Let $C^i(S,c) = \{ k : [0, a^i(S,c)] \to R_+ \,/\, k(0) = 0,$ k is concave, continuous and increasing$\}$

for $i \in M$. Here for $i \in M$, $a^i(S,c) = \max \{c_i, u_i(S)\}$

Where $u_i(S) = \max_{x \in S} x_i$

For $(S,c) \in D$, $i \in M$, $k^i \in C^i(S,c)$, let $(k^i(S), k^i(c))$ be defined as follows :

$k^i(c) \in R^M_+$ with $k^i_j(c) = c_j \; \forall j \neq i$, $k^i_i(c) = k^i(c_i)$

$k^i(S) = \{ y \in R^M_+ \,/\, y_j = x_j$ if $j \neq i$, $y_i = k^i(x_i)$, $x \in S \}$

Clearly $(k^i(S), k^i(C)) \in D$.

We shall now show that $F : \mathcal{L}(M) \to R^M_+$ satisfies the following property $\forall (S,c) \in D$, $k^i(F_i(S,c)) \geq F_i(k^i(S), k^i(c))$ whenever $i \in M$ and $k^i \in C^i(S,c)$

Infact by Axiom 2, we may assume $k^i(c_i) = c_i$

Suppose towards a contradiction $k^i(F_i(S,c)) < F_i(k^i(S),c)$

Now $F_i(S,c) < c_i$ and $k^i(c_i) = c_i$, $k^i(0) = (0)$ and k^i concave implies $k^i(F_i(S,c)) \geq F_i(S,C)$.

Therefore

$$\frac{F^i(k^i(S),c)}{c_i} > \frac{k^i(F_i(S,c))}{c_i} \geq \frac{F_i(S,c)}{c_i}$$

$$\frac{P_j(k^i(S),c)}{c_j} > \frac{P_j(S,c)}{c_j} \qquad \forall\, j \neq i.$$

Hence $P_j(k^i(S), c) > P_j(S,c) \qquad \forall\, j \neq i.$

But k^i increasing implies $(k^i)^{-1}(P_i(k^i(S),c)) > P_i(S,c)$.

Further $((k^i)^{-1}(P_i(k^i(S), c)), P_{-i}(k^i(S),c)) \in S$.

This contradicts the fact that P satisfies efficiency.

Hence $k^i(P_i(S,c)) \geq P_i(k^i(S), c)$

Thus the proportional solution satisfies what many be called improvement sensitivity on a somewhat restricted and yet reasonable sub-domain of $\mathcal{B}(M)$.

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