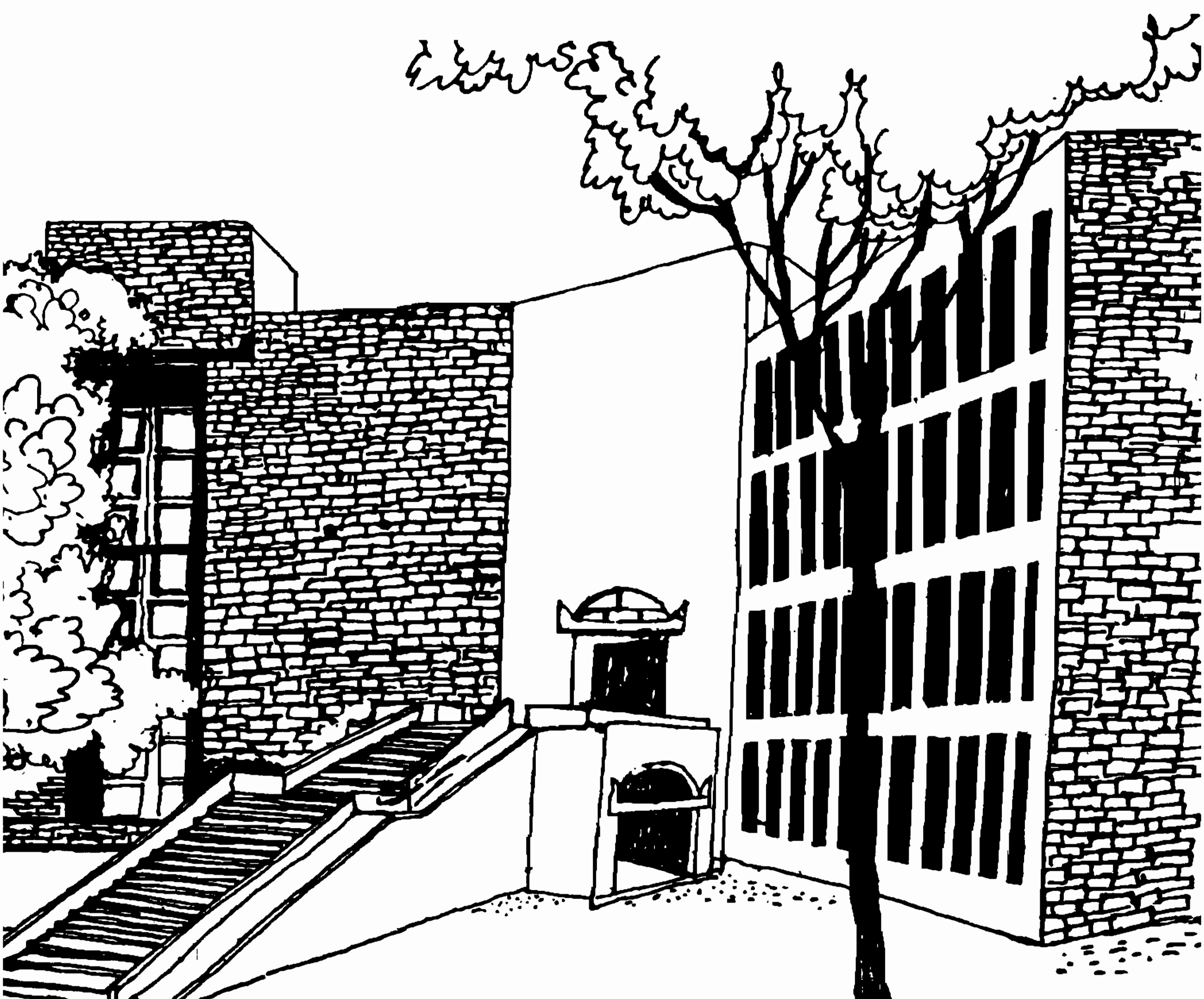
Working Paper



AN AXIOMATIC CHARACTERIZATION OF THE EQUAL LOSS CHOICE FUNCTION FOR MULTIATTRIBUTE CHOICE PROBLEMS

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Abstract

In this paper we provide an axiomatic characterization of the equal loss choice function for multiattribute choice problems.

1. Introduction :- In Yu (1985), can be found the beginnings of a theory of multiattribute choice problems and a statement of the equal loss choice function for such problems. There a number of properties of this and other compromise solutions has been discussed. In Lahiri (1993b) the framework was partially extended to study a certain "monotonicity with respect to the target point" property of the entire family of compromise solutions suggested by Yu. In Chun and Thomson (1992) and Lahiri (1993a) different axiomatic characterizations have been provided for a different choice function to multiattribute choice problems which satisfies a property called "restricted monotonicity with respect to the target point". An axiomatic characterization of a choice function is a statement of some properties which uniquely characterizes the choice function. In this respect the earliest axiomatic characterization of the equal loss solution is due to Chun (1988). Subsequently Bossert (1992) provided a different axiomatic characterization of the same solution. However, both these studies considered domains which admitted unbounded choice problems.

In this paper, we consider choice problems which are bounded both above and below. This is more in keeping with the spirit of multiattribute choice theory as enunciated for instance in Keeney and Raiffa (1976) and Yu (1985). As a result of this modification, the earlier characterization results break down and what replaces it is completely original both in content and style. Application of this choice theory to production planning problems can be found in Abad and Lahiri (1993) and Lahiri (1993a).

2. Multiattribute Choice Problems :- A multiattribute choice problem is an ordered pair (S,c) where $o \in S \in \mathbf{R}^n_+$ and $c \in \mathbf{R}^n_+$ for some $n \in \mathbf{N}$ (the set of natural numbers). The set S is called the set of feasible attribute vectors and the point c is called a target point.

We shall consider the following class $\mathcal{Q}$ of admissible multiattribute choice problems: $(S,c)\in\mathcal{L}$ if and only if

(i) S is compact and convex

(ii) S satisfies minimal transferability: $x\in S$, $x_i>0$ $\Rightarrow \exists y\in S$ with $y_i<x_i$ and $y_j>x_j\ \forall j\neq i$.

(iii) S is comprehensive: $x\in S$, $0\leqq y\leqq x \Rightarrow y\in S$.

(Here for $x,y\in\mathbb{R}^n$, $x\geqq y$ means $x_i\geqq y_i\ \forall i\in\{1,\ldots,n\}$; $x>y$ means $x\geqq y$ but $x\neq y$; $x\gg y$ means $x_i>y_i\ \forall i=1,\ldots,n$).

The structural isomorphism of $\mathcal{L}$ to the class of production planning problems with quasi-convex cost functions has been discussed in Abad and Lahiri (1993).

A <u>domain</u> is any subset D of $\mathcal{L}$.

A (<u>multiattribute</u>) <u>choice function</u> on D is a function $F:D\to\mathbb{R}^n_+$ such that $\forall(S,c)\in D$, $F(S,c)\in S$.

Let $F:D\to\mathbb{R}^n_+$ be a choice function. Three important properties often required of a choice function are the following:

(P.1) <u>Efficiency</u> :- $\forall(S,c)\in D$, $x\in S$, $x\geqq F(S,c) \Rightarrow x=F(S,c)$.

(P.2) <u>Symmetry</u> :- If $\forall$ permutation $\sigma:N\to N$, $\sigma(S)=S$ and $\sigma(c)=c$, then $F_i(S,c)=F_j(S,c)\ \forall i,\ j\in\{1,\ldots,n\}$.

Here for $x\in\mathbb{R}^n_+$, $\sigma(x)$ **is the vector in** $\mathbb{R}^n_+$ whose ith coordinate is $x_{\sigma(i)}$ **and** $\sigma(S)=\{\sigma(x):x\in S\}$.

(P.3) <u>Restricted Monotonicity</u> :- $\forall(S,c)$, $(S',c)\in\mathcal{L}$, $S\subseteq S'$ $\Rightarrow F(S,c)\leqq F(S',c)$.

In order to define the equal loss choice function we consider the following domain:

$$D\equiv\{(S,c)\in\mathcal{L}/c-[\min_i(c_i)]e\in S\}$$

where e is the vector in $\mathbb{R}^n$ with all coordinatse equal to one.

The equal loss choice function $Y^*:D\to\mathbb{R}^n_+$ is defined as follows:

$$Y^*(S,c)=c-\bar{\lambda}e,\text{ where }\bar{\lambda}=\min\{\lambda\in\mathbb{R}/c-\lambda e\in S\}$$

It is easy to see that Y^* is well defined. This solution is originally due to P.L. Yu.

In order to characterize Y^* axiomatically we will require the following property:

Let $(S,c)\in D$ and $a\in\mathbb{R}^n_+$. Then if $a\leqq c$ and $(S-\{a\})\cap\mathbb{R}^n_+\neq\emptyset$, we have the choice problem $(S(a), c-a) \in D$ where $S(a) = (S-\{a\}) \cap \mathbb{R}^n_+$.

(P.4) <u>c-Shift Invariance</u> :- $\forall (S,c) \in D$ $\forall a \in \mathbb{R}^n_+$ such that $a \leqq c - [\min_i (c_i)]e$,

$F(S(a), c-a) = F(S,c) - a$.

In the next section we show that the above four properties characterize Y^* on D.

3. The Main Theorem :-

<u>Theorem</u> :- The only choice function on D to satisfy properties (P.1), (P.2), (P.3) and (P.4) is the equal loss choice function.

<u>Proof</u> :- It is easily verified that Y^* satisfies the above properties. Hence let $F: D \to \mathbb{R}^n_+$ be any choice function satisfying the above properties. If $S = \{0\}$, then by the definition of a choice function $F(S,c) = 0 = Y^*(S,c)$. Hence suppose $S \neq \{0\}$ and let $a = c - [\min_i (c_i)]e$. Then $Y^*(S(a), c-a) = \bar{\lambda}.e$ for some $\bar{\lambda} \geqq 0$. If $S(a) = \{0\}$, then $\bar{\lambda} = 0$ and $F(S(a), c-a) = 0$, so that by appealing to (P.4) we get $F(S,c) = Y^*(S,c)$. Hence suppose $S(a) \neq 0$, so that $\bar{\lambda} > 0$. By minimal transferability, $\forall i \in \{1,...,n\}$, there exists $v^i \in S(a)$, such that $v^i_i < \bar{\lambda}$ and $v^i_j > \bar{\lambda}$ if $j \neq i$. Let $\alpha = \min_{1 \leqq i \neq j \leqq n} v^i_j$. Clearly $\alpha > \bar{\lambda}$. For $i \in \{1,...,n\}$, let $a^i \in \mathbb{R}^n_+$ such that $a^i_i = 0$, $a^i_j = \alpha$ for $j \neq i$. Clearly $a^i < v^i$ and by comprehensiveness, $a^i \in S$ $\forall i \in \{1,...,n\}$. Let $T = \text{convex hull } \{0, a^1, ..., a^n, \bar{\lambda}e\}$. T is symmetric, $\bar{\lambda}e$ is efficient in T, $\bar{\lambda}e$ has all coordinates equal to $\bar{\lambda} > 0$ and c-a has all coordinates equal to $\min_i (c_i)$. Hence by (P.2) and (P.1), $F(T, c-a) = \bar{\lambda}e$. Now, $T \subseteq S(a)$. Hence by (P.3) $F(S(a), c-a) \geqq F(T, c-a) = \bar{\lambda}e$. However $\bar{\lambda}e = Y^*(S(a), c-a)$ is an efficient point in S(a). Thus $F(S(a), c-a) = \bar{\lambda}.e = Y^*(S(a), c-a)$. By (P.4) applied to both F and Y^* we get since $a \leqq c - [\min_i (c_i)]e$, $F(S,c) = Y^*(S,c)$.

Q.E.D.

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