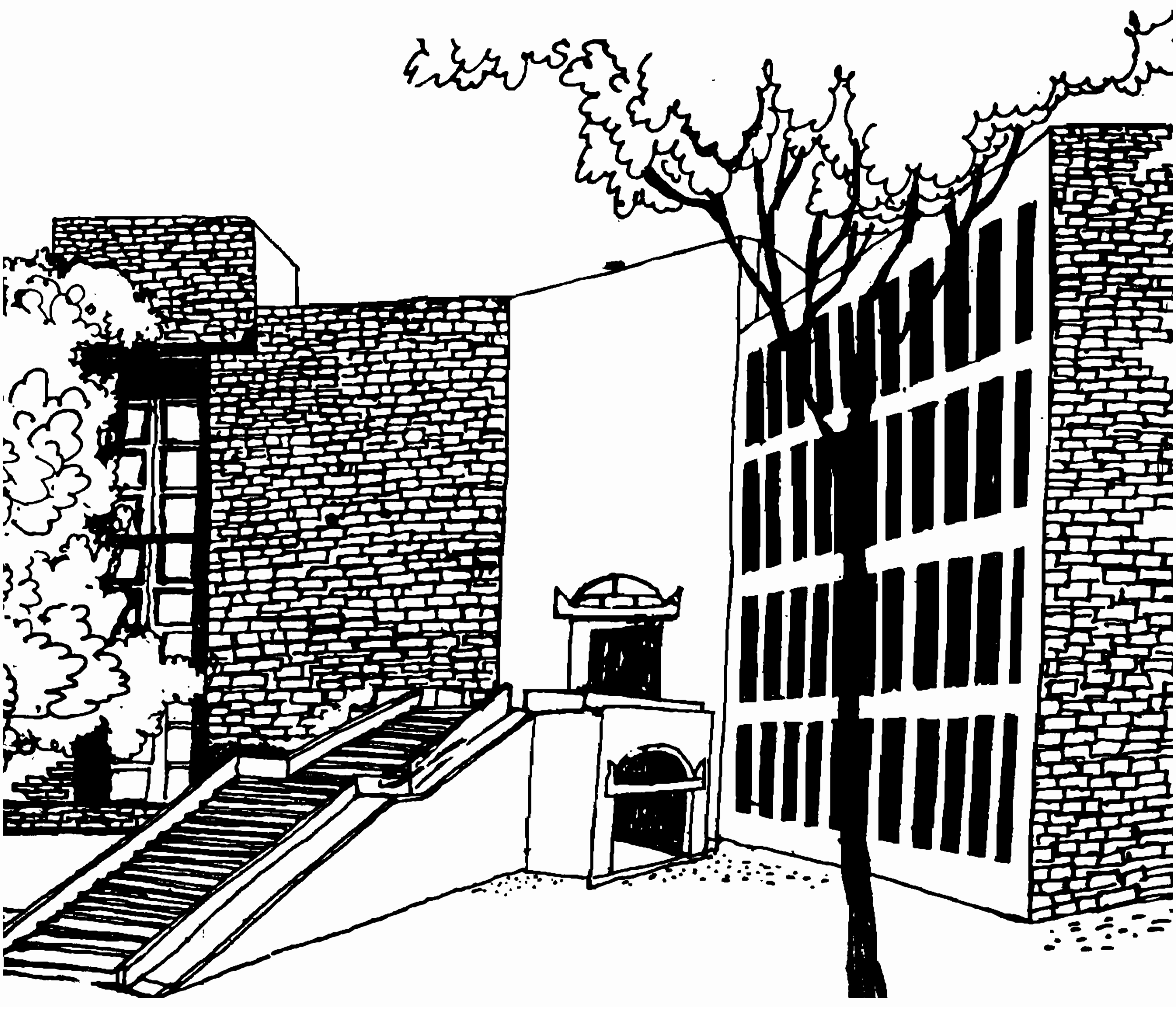
विद्याविनियोगाद्विकासः

IIM AHMEDABAD

Working Paper



ON A THEOREM DUE TO SOBEL

By

Somdeb Lahiri

W P No. 1131
August 1993

The main objective of the working paper series of the IIMA is to help faculty members to test out their research findings at the pre-publication stage.

INDIAN INSTITUTE OF MANAGEMENT
AHMEDABAD-380 015
INDIA

Abstract

In this paper we prove that any bargaining solution to group decision problems which satisfies individual rationality, strong symmetry, efficiency and strong improvement sensitivity also satisfies mid-point domination.

1. Introduction :- In Sobel (1981) can be found a result in two attribute choice theory, which states that if a choice function satisfies efficiency and risk-sensitivity then it satisfies mid-point domination. Using this result one may assert that the Nash (1950) as well as the relative egalitarian solution due to Kalai and Smorodinsky (1975) **satisfy mid-point domination.**

In Peters (1992), the concept of risk sensitivity has been modified to improvement sensitivity, in order to accommodate group decision problems under certainty, in the analysis, and Lahiri (1993) extends this analysis to group decision making problems with claims. In this paper we show that the result due to Sobel can be extended to problems studied in Lahiri (1993). Of course in order to do so, we use instead of risk sensitivity the more appropriate concept of strong improvement sensitivity.

2. Group Decision Problems With Claims :- Our framework of analysis is adopted from Lahiri (1993).

An <u>n-person group decision making problem with claims</u> Ω has the form $\langle A,\bar{a},u^1,\ldots,u^n,v\rangle$ where

(i) A is non-empty, compact, convex set in $\mathbb{R}^k$ for some $k\in\mathbb{N}$. A is called the set of alternatives

(ii) $\forall\ i\in\{1,\ldots,n\}$, $u^i:A\to\mathbb{R}$ is a continuous and concave (value) function

(iii) $\bar{a}\in A$ is a status-quo alternative

(iv) $v=(v_1,\ldots,v_n)\in\mathbb{R}^n$, $v_i\geq u_i(a)\ \forall i=1,\ldots,n$. v is called the claims point.

We denote by D the family of all group decision making problems with claims.

A domain is any subset $\mathcal{Q}$ of D.

A <u>solution</u> on $\mathcal{Q}$ in a non-empty valued correspondence

$F:\mathcal{Q}\to\bigcup A$ such that for each $\Omega\in\mathcal{Q}$
$A: \Omega=\langle A,\bar{a},u^1,\ldots,u^n,v\rangle\in\mathcal{Q}$

$\Omega=\langle A,\bar{a},u^1,\ldots,u^n,v\rangle, F(\Omega)\subseteq A.$

A solution F on $\mathcal{Q}$ is said to be <u>efficient</u> or <u>Pareto-optimal</u> if $\forall\Omega\in\mathcal{Q}$, $\Omega=\langle A,\bar{a},u^1,\ldots,u^n,v\rangle$, $a\in F(\Omega)$, $b\in A$, $u^i(b)\geq u^i$

$(a)\forall i=1,\ldots,n \Rightarrow u^i(b)=u^i(a)\ \forall i=1,\ldots,n.$

A solution F on $\mathcal{Q}$ is said to be individually rational if $\forall\ \Omega\in\mathcal{Q}$, $\Omega=\langle A,\bar{a},u^1,\ldots,u^n,v\rangle$, $a\in F(\Omega)\Rightarrow u^i(a)\geq u^i(\bar{a})\ \forall i=1,\ldots,n.$

A solution F on $\mathcal{Q}$ is said to satisfy scale covariance if $\forall\alpha\in\mathbb{R}^n_{++}$, $\beta\in\mathbb{R}^n$, $\Omega\in\mathcal{Q}$, $\Omega=\langle A,\bar{a},u^1,\ldots,u^n,v\rangle$, $\Omega'=\langle A,\bar{a},\alpha_1 u^1+\beta_1,\ldots,\alpha_n u^n+\beta_n,\alpha_1 v_1+\beta_1,\ldots,\alpha_n v_n+\beta_n\rangle\in\mathcal{Q}$, then $F(\Omega)=F(\Omega')$.

A domain which is closed under the operation required in the definition of scale covariance is called a scale covariant domain.

A domain $\mathcal{Q}$ is said to be improvement invariant if $\forall\ \Omega=\langle A,\bar{a},u^1,\ldots,u^n,v\rangle\ \forall i\in\{1,\ldots,n\}\ \forall k:[\min_{a\in A} u^i(a),+\infty)\rightarrow\mathbb{R}$ which is concave, continuous and strictly increasing, $\Omega'=\langle A,\bar{a},u^1,\ldots,u^{i-1},k\circ u^i,u^{i+1},\ldots,u^n,v^1,\ldots,v^{i-1},k(v^i),v^{i+1},\ldots,v^n\rangle\in\mathcal{Q}$.

Let $\mathcal{Q}$ be a domain and F a solution on $\mathcal{Q}$. F is said to be a bargaining solution on $\mathcal{Q}$ if $\forall\ \Omega=\langle A,a,u^1,\ldots,u^n,v\rangle\in\mathcal{Q}$, $a\in F(\Omega), b\in A, [u^i(a)=u^i(b)\forall i=1,\ldots,n \Leftrightarrow b\in F(\Omega)]$. It should be noted that a bargaining solution is strictly more general than the welfarist n-person bargaining solutions studied in axiomatic models of bargaining. This has been discussed in Lahiri (1993).

For reasons discussed in Peters (1992), we say that a bargaining solution F defined on an improvement invariant domain $\mathcal{Q}$ is strongly improvement sensitive if $\forall\ \Omega=\langle A,\bar{a},u^1,\ldots,u^n,v\rangle\forall i\in\{1,\ldots,n\}\forall k$ as defined above and Ω' as defined above, $u^j(a)\geq u^j(b)\forall a\in F(\Omega), b\in F(\Omega')$ and $\forall j\neq i, j\in\{1,\ldots n\}$.

It is easy to see that our definition of strong improvement sensitivity implies the concept of improvement sensitivity of Peters (1992), if we require in addition that the solution is efficient. For the case of n=2, the two definitions are equivalent for efficient solutions.

Given $\Omega\in\mathcal{Q}$, let $P(\Omega)=\{u(a)/a\in A$ and $b\in A$ with $u(b)\geq u(a)\Rightarrow u(b)=u(a)\}$

$IR(\Omega)\equiv\{u(a)/a\in A$ and $u(a)\geq u(\bar{a})\}$ where $\Omega\equiv\langle A,\bar{a},u^1,\ldots,u^n,v\rangle$ and $u=(u^1,\ldots,u^n)$. The above two sets are respectively the efficient set of Ω and the individually rational set of Ω.

For the subsequent analysis we also require the following property:

F on $\mathcal{Q}$ satisfies <u>strong symmetry</u> if given $\Omega=\langle A,\bar{a},u,v\rangle \in \mathcal{Q}$ if $\forall$ $\sigma:\{1,\ldots,n\}\to\{1,\ldots,n\}$ which are one-to-one $\sigma(IR(\Omega))=IR(\Omega)$, then $u^i(F(\Omega))=u^j(F(\Omega))\forall i,j\in\{1,\ldots,n\}$. Here for any $x\in\mathbb{R}^n$, $\sigma(x)$ is the vector with $\sigma(x)_i=x_{\sigma(i)}$, and $\sigma(S)=\{\sigma(x):x\in S\}$ for any subset S of $\mathbb{R}^n$.

2. A Preliminary Result on Scale Covariant domains :-

<u>Lemma 1</u> :- Let $\mathcal{Q}$ be a scale covariant domain and F be a bargaining solution on $\mathcal{Q}$ which satisfies efficiency and strong improvement sensitivity. Then F satisfies scale covariance.

<u>Proof</u> :- It is enough to prove that if $i\in\{1,\ldots,n\}$, $\alpha_i>0$, $\beta_i\in\mathbb{R}$, $\Omega=\langle A,\bar{a},u^1,\ldots,u^n,v\rangle$, $\Omega'=\langle A,\bar{a},u^1,\ldots,u^{i-1},\alpha_i u^i+\beta_i,u^{i+1},\ldots,u^n,v^1,\ldots,v^{i-1},\alpha_i v_i+\beta_i,v^{i+1},\ldots,v^n\rangle$ then $F(\Omega)=F(\Omega')$. Now for the given $\alpha_i>0,\beta_i\in\mathbb{R}$, Ω' is obtained from Ω by an increasing, continuous, concave (in fact affine transformation) of u^i and v^i. Hence by strong improvement sensitivity, $\forall j\neq i$ $u^j(F(\Omega))\leq u^j(F(\Omega'))$. However the inverse transformation also satisfies identical properties and hence $u^j(F(\Omega'))\leq u^j(F(\Omega))$. Thus $u^j(F(\Omega'))=u^j(F(\Omega))\forall j\neq i,j\in\{1,\ldots,n\}$. Hence by efficiency of F and the fact that $\alpha_i>0$, we get $u^i(F(\Omega'))=u^i(F(\Omega))$. Since F is a bargaining solution, we must have $F(\Omega')=F(\Omega)$.

Q.E.D.

The above proof is essentially similar to that of Kihlstrom, Roth and Schmeidler (1981). However the scope of our theorem extends beyond that of the earlier theorem since our domains and solutions are far more general.

3. Sobel's Theorem for Two Person Group Decision making Problems With Claims :-

Sobel (1981) had proved the result that we discussed in the Introduction, for choice problems with two attributes. We prove a similar result for two person group decision making problems. Our result is more general, since our solutions are neither welfarist, nor is the problem restricted to dividing a fixed amount of resource between two agents. However to facilitate presentation we make the following assumptions, similar to those appearing in Cornet (1976).

Assumption 1 :- If $\Omega \equiv \langle A, \bar{a}, u, v\rangle \in \mathcal{Q}$, then $u^i : A \to \mathbb{R}$ is strictly concave for $i=1,2$.

Assumption 2 :- If $IR(\Omega)$ has nonempty interior $\forall\ \Omega \in \mathcal{Q}$.

Assumption 3 :- If $\sigma(IR(\Omega)) = IR(\Omega)$ for $\sigma : \{1,2\} \to \{1,2\}$ with $\sigma(1)=2$, $\sigma(2)=1$, then $v_1 = v_2$ where $\Omega \equiv \langle A, \bar{a}, u, v\rangle$

Assumption 4 :- $\mathcal{Q}$ is a scale covariant, improvement invariant domain.

Theorem :- Let F be a bargaining solution on $\mathcal{Q}$ satisfying individual rationality, strong symmetry, efficiency and strong improvement sensitivity. Then

$$\forall\ i \in \{1,2\},\ u^i(F(\Omega)) \geqq \max_{a \in IR(\Omega)} \frac{u^i(a)}{2}\ \ \forall\ \Omega = \langle A, \bar{a}, u, v\rangle$$

(F satisfies midpoint domination).

Proof :- Since efficiency and strong improvement sensitivity implies scale covariance, we can assume that $\max_{a \in IR(\Omega)} u^i(a) = e^i$ where e^i is the ith co-ordinate vector in $\mathbb{R}^2$ and $u(\bar{a})=0$. As in Sobel (1981), there exists a continuous, convex, increasing function $k : \mathbb{R}_+ \to \mathbb{R}_+$ such that if $\Omega' = \langle A, \bar{a}, k o u^1, u^2, k(v^1), v^2\rangle$ then $IR(\Omega') = \{(x_1, x_2) \in \mathbb{R}^2 / x_1 + x_2 \leqq 1\}$. By efficiency and strong symmetry, $u^1(F(\Omega')) = u^2(F(\Omega')) = \frac{1}{2}$. Since the inverse of a continuous, convex, increasing function from $\mathbb{R}_+$ to $\mathbb{R}_+$ is a continuous, concave increasing function, by strong improvement sensitivity, $u^2(F(\Omega)) \geqq u^2(F(\Omega')) = \frac{1}{2}$. By an identical argument we can show that $u^1(F(\Omega)) \geq \frac{1}{2}$. Hence F satisfies mid-point domination.

Q.E.D.

4. Conclusion :- It is important to notice that our result generalizes a similar result proved earlier, since our solutions are not required to be welfarist i.e. the solution does not depend only on the feasible utility vectors. The proofs however are similar, highlighting the larger scope of the literature on axiomatic models of bargaining than to what it has been restricted to so far. Further, our domain is larger than the domain studied in axiomatic bargaining. What we actually show is that similar results continue to hold in group decision making problems using assumptions which are weaker than earlier ones.

Appendix

In this appendix we show if $\Omega = \langle A, \bar{a}, u, v \rangle \in \mathcal{L}$ and satisfy the assumptions of Section 3, then there exists a convex, continuous and increasing function $k: \mathbb{R}_+ \to \mathbb{R}_+$ such that if $\Omega' = \{A, \bar{a}, k \circ u^1, u^2, k(v_1), v_2\}$ then $IR(\Omega') = \{x \in \mathbb{R}^2_+ / x_1 + x_2 \leq 1\}$. We shall assume without loss of generality that $u(\bar{a}) = 0$ and $\max\{x_i / x \in IR(\Omega)\} = 1$ for $i = 1, 2$.

Let $\emptyset: [0,1] \to [0,1]$ be such that $\{(y, \phi(y)) / y \in [0,1]\} = P(\Omega) \cap IR(\Omega)$. Clearly $\emptyset$ is decreasing and concave. Further $\emptyset(0)=1, \emptyset(1)=0$. Let $h(y)=\phi(1-a)$. Then $h:[0,1] \to [0,1]$ is increasing and concave and satisfies $h(0)=0$, $h(1)=1$. Let $k:[0,1] \to [0,1]$ be defined by $k=h^{-1}$. k is increasing and convex, and is the desired transformation.

References :-

1. B. Cornet (1976) : "On planning procedures defined by multivalued differential equations", in Proceedings of the CNRS International Colloquium: "Systems dynamiiques et Modeles economiques".

2. E. Kalai and M. Smorodinsky (1975) : "Other Solutions to Nash's Bargaining Problem", Econometrica 43, 513-518.

3. R.E. Kihlstrom, A.E. Roth and D. Schmeidler (1981) : "Risk Aversion and Solutions to Nash's Bargaining Problem", in Game Theory and Mathematical Economics (O. Moeschlin and D. Pallaschke, eds), North-Holland, 65-71.

4. S. Lahiri (1993) : "Improvement Sensitivity And Invariance of Solutions To Group Decision Making Problems With Claims", I.I.M.A. W.P.No. 1116.

5. J. Nash (1950) : "The Bargaining Problem", Econometrica 18, 155-162.

6. H. Peters (1992) : "A Criterion for Comparing Strength of Preference With An Application To Bargaining", Operations Research, Vol. 40, No. 5, 1018-1022.

7. J. Sobel (1981) : "Distortion Of Utilities And The Bargaining Problem", Econometrica 49, 597-619.