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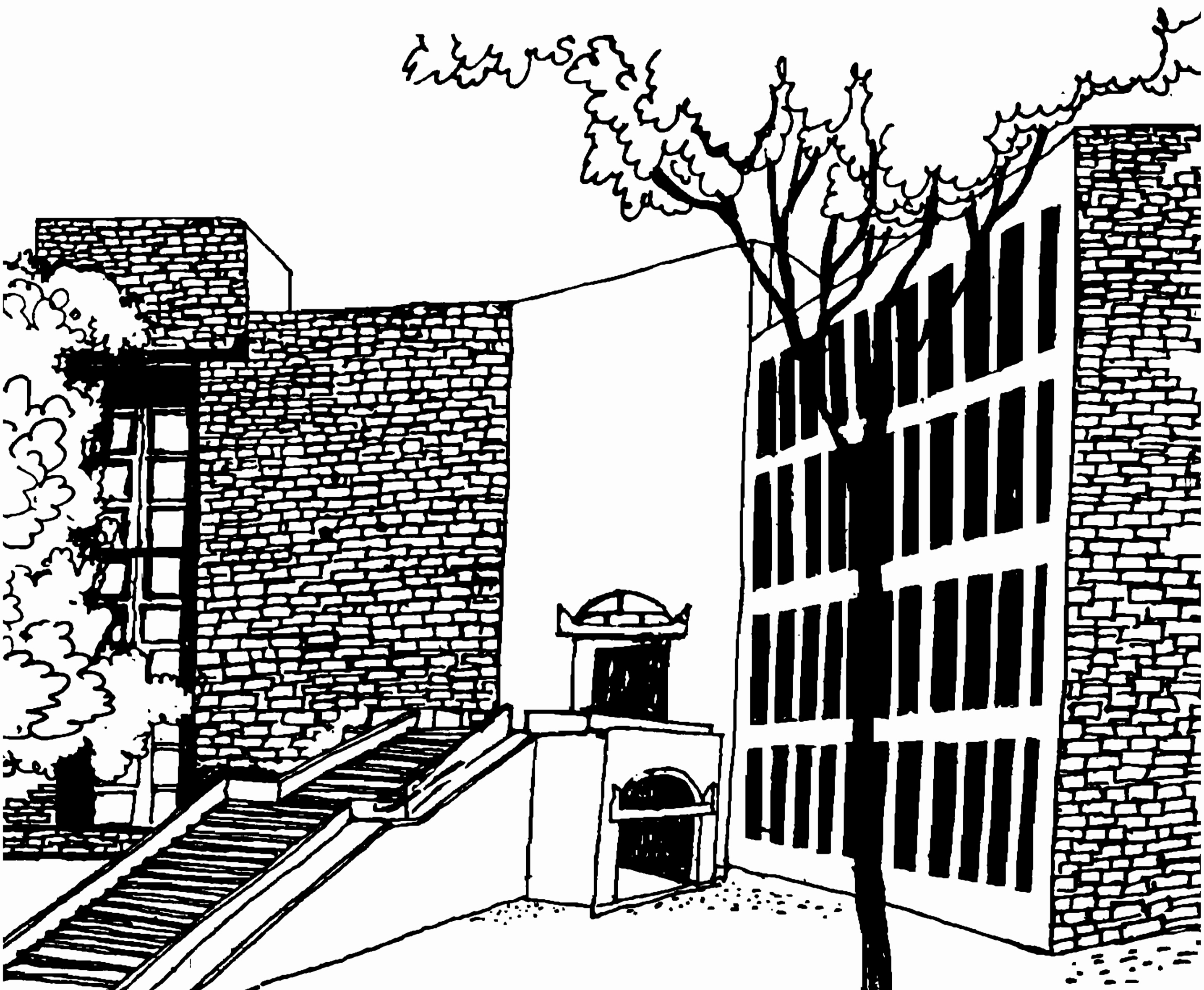

ON THE EXISTENCE AND EFFICIENCY OF A VOTING EQUILIBRIUM FOR A PUBLIC GOOD ECONOMY

By

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Abstract

The main purpose of this paper is to formalise the concept of a compromise function. Subsequently, given a compromise function, we define a voting equilibrium and prove the existence of such an equilibrium. Finally, we close our analysis by showing that under some assumptions a voting equilibrium is Pareto optimal.

1. Introduction :- The classical theory of resource allocation for the provision of a public good, concerns itself with each consumer in the economy being presented with a price for the public good (possibly different), so that no one has any incentive to make a unilateral choice distinct from the others. Given the nonexcludable nature of public consumption, differences in prices (personalized prices) are used to guarantee unanimity of choice. Such is the spirit behind the more well known solutions to problems of decentralized choice of public expenditure as in Foley (1970) or Kaneko (1977). The solution concept analysed by Foley is the Lindahl equilibrium solution. Kaneko proposed and analysed the ratio equilibrium solution.

However, the fact that public consumption must be equal for everyone, does not automatically imply that everyone will ask for the same level of public expenditure. In fact, it is quite likely that different beneficiaries of the public good, left to themselves, would ask for different levels of consumption of the good. The solution concepts mentioned above, use personalized prices to make consumers agree. However, it is quite reasonable, as insisted upon by Aczel (1987) (Section 1) that society requires a rule by which to arrive at a compromise solution for divergent proposals. To some extent, we believe, that this is precisely the method by which actual public projects are realized in practice. Society or better still a social decision maker intervenes, to arrive at a compromise solution for the public good consumption level on the basis of a list of such proposals obtained from individual consumers. As far as deciding the level of public expenditure goes, "the external arbitrator" (which for most other purposes is only a convenient fiction) plays a more active and significant role than what received theory would have us believe.

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2. The Model :- Our framework of analysis draws on Lahiri (1993). Thus we assume that our economy consists of two goods: (a) a private good, which is both a numeraire, as well as an input for all production that takes place in the economy; (b) a public good, which is produced using the private good and whose final consumption level is the same for all individuals in the economy and is also equal to its level of production. We assume that there are $n \geq 2$ individuals in the economy. Let $w_i > 0$ be the initial endowment of the private good with the ith individual. The cost function for the production of the public good from the private good is $c: \mathbb{R}_+ \to \mathbb{R}_+$ satisfying (i) $c(0)=0$; (ii) c is strictly increasing; (iii) c is continuous. (Here $\mathbb{R}_+$ is the set of all non-negative real numbers.) Sometimes we shall also assume that (iv) c is a convex function.

The preferences of each individual i is represented by a function $u_i : \mathbb{R}^2_+ \to \mathbb{R}$ where if $x_i \geq 0$ denotes his consumption of the private good and $y \geq 0$ denotes society's consumption of the public good, then $u_i(x_i, y)$ denotes individual i's utility derived from the consumption bundle $(x_i, y) \in \mathbb{R}^2_+$. (Here $\mathbb{R}^2_+ \equiv \mathbb{R}_+ \times \mathbb{R}_+$). We assume that:

(v) u_i is continuous

(vi) u_i is strongly monotonic i.e. $(x_i, y) \geq (x'_i, y'), (x_i, y) \neq (x'_i, y') \Rightarrow u_i(x_i, y) > u_i(x'_i, y')$.

In addition, we sometimes assume

(vii) u_i is quasi-concave i.e. $\forall (x_i, y), (x'_i, y') \in \mathbb{R}^2_+$, $\forall \alpha \in [0,1]$, $u_i(\alpha x_i + (1-\alpha)x'_i, \alpha y + (1-\alpha)y') \geq \min\{u_i(x_i, y), u_i(x'_i, y')\}$.

We now define a compromise function. A function $G: \mathbb{R}^n_+ \to \mathbb{R}$ which is continuous and strictly increasing (i.e. $\forall y, y' \in \mathbb{R}^n_+$, $y \geq y', y \neq y' \Rightarrow G(y) > G(y')$) is called a compromise function. The interpretation of a compromise function is that corresponding to a set of proposals for public good consumption made by the individuals. G recommends a comrpomise solution by a social decision maker. (Here, $\mathbb{R}^n_+$ denotes the n-fold Cartesian product of $\mathbb{R}_+$; $\forall y, y' \in \mathbb{R}^n_+$, $y=(y_i)^n_{i=1}$, $y'=(y'_i)^n_{i=1}$, $[y \geqq y' \iff y_i \geqq y'_i$ $\forall i=1,\ldots,n)$.

In the sequel the following notation will be found convenient: $\forall y \in \mathbb{R}^n_+$, $\forall i \in \{1,\ldots,n\}$, $y_{-i} = (y_1,\ldots,y_{i-1},y_{i+1},y_n)$ and $y \equiv (y_i, y_{-i})$.

A voting equilibrium for the above economy is an ordered pair (y^*, t^*):

(i) $t^* \in \{t \in \mathbb{R}^n_+ / \Sigma^n_{i=1} t_i = 1\} \equiv \Delta^{n-1}$ (the n-1 dimensional simplex).

(ii) $y^*_i = \arg\max\limits_{y_i \geqq 0} u_i(w_i - t^*_i\, c(G(y_i, y^*_{-i})), G(y_i, y^*_{-i}))$

$w_i - t^*_i\, c(G(y_i, y^*_{-i})) \geqq 0$

$\forall i=1,\ldots,n.$

The close relation between a voting equilibrium and Kaneko's ratio equilibrium cannot be missed. A voting equilibrium consists of a set of personalized prices and a set of public good consumption demands, such that the corresponding compromise solution satisfies the budget equation of all individuals and no single individual can benefit by unilaterally deviating to any other public good consumption demand which along with the consumption demands of the others yields an affordable compromise solution. In the subsequent section we show that under some conditions a voting equilibrium exists.

3. Existence of Voting Equilibrium :- Throught this section we assume that $c:\mathbb{R}_+ \to \mathbb{R}_+$ satisfies properties (i) to (iv) listed in section 2 and $\forall i=1,\ldots,n$, $u_i : \mathbb{R}_+ \to \mathbb{R}_+$ satisfies properties (vi) to (viii) listed in section 2. It is easy to show that under properties (vi) to (viii), $u_i : \mathbb{R}^2_+ \to \mathbb{R}_+$ satisfies <u>semi-strict quasi-concavity</u> i.e. $(x_i, y), (x'_i, y') \in \mathbb{R}^2_+$, $u_i(x_i, y) > u_i(x'_i, y') \Rightarrow u_i(tx_i + (1-t)x'_i, ty+(1-t)y') > u_i(x'_i, y')\ \forall t \in (0,1]$. <u>In addition we assume that the compromise solution G is linear</u>.

Let $F=\{y \in \mathbb{R}^n_+ / c(G(y)) \leqq \Sigma^n_{i=1} w_i\}$. Then is a convex set. Let $F'=\{y \in \mathbb{R}^n_+ / c(G(y)) \leqq \Sigma^n_{i=1} w_i + 1\}$. It is easy to see that F' is a convex set. Let S be an n-dimensional symmetric cube with centre at 0, containing F'.

The following lemma essentially due to Kaneko (1977) will be stated without proof.

Lemma 1 :- Given $y^*_{-i} \in \mathbb{R}^{n-1}_+$, $t^*_i > 0$, let y^*_i solve :

$$u_i(w_i - t^*_i c(G(y_i, y^*_{-i})), G(y_i, y^*_{-i})) \to \max$$
$$\text{s.t. } y_i \geqq 0$$
$$w_i - t^*_i c(G(y_i, y^*_{-i})) \geqq 0$$
$$(y_i, y^*_{-i}) \in F'$$

Then if $y^* \in F$, y^*_i solves:

$$u_i(w_i - t^*_i c(G(y_i, y^*_{-i})), G(y_i, y^*_{-i})) \to \max$$
$$\text{s.t. } y_i \geqq 0$$
$$w_i - t^*_i c(G(y_i, y^*_{-i})) \geqq 0.$$

For our subsequent analysis let us define $\forall i=1,\ldots,n$, f_i : $\Delta^{n-1} \times S \rightarrow\rightarrow \mathbb{R}_+$ as follows: $\bar{g}_i \in f_i(t,y)$ if and only if $\bar{g}_i$ solves:

$$u_i(w_i - t_i c(G(g_i, y_{-i})), G(g_i, y_{-i})) \to \max$$
$$\text{s.t. } g_i \geqq 0$$
$$w_i - t_i c(G(g_i, y_{-i})) \geqq 0,$$
$$(g_i, y_{-i}) \in F'.$$

Assumption :- $\forall y \in S$, $\forall i \in \{1,\ldots,n\}$, $\exists$ $g_i \geqq 0$ such that $c(G(g_i, y_{-i})) \leqq w_i$.

This assumption along with the ones made above guarantees that the correspondence f_i is non-empty valued $\forall i=1,\ldots,n$. It is easy to see that f_i is convex valued, convex-valued and upper-semicontinuous (see Hildenbrand and Kirman (1988)).

Theorem 1 :- Under the assumptions invoked in this section, there exists $(y^*, t^*) \in \mathbb{R}^n_+ \times \Delta^{n-1}$, such that the pair forms a voting equilibrium.

Proof :- Consider the correspondence h : $\Delta^{n-1} \times S \rightarrow\rightarrow \Delta^{n-1} \times S$ defined as follows:

$$h(t,g) = \{t\} \times \pi^n_{i=1} f_i(t,g)$$

It is easy to verify that h is well-defined, non-empty valued, convex-valued, compact-valued and upper-semicontinuous. Thus by Kakutani's fixed point theorem, there exists $(t^*, y^*) \in$

$\Delta^{n-1} \times S$ such that $(t^*, y^*) \in h(t^*, y^*)$. By Lemma 1, (t^*, y^*) is a voting equilibrium.

Q.E.D.

4. The Pareto Optimality of a voting equilibrium with quasi-linear preferences :- In the general case, a voting equilibrium allocation need not be Pareto Optimal, where an allocation $(x, g) \in \mathbb{R}^n_+ \times \mathbb{R}_+$ such that $\Sigma^n_{i=1} x_i + c(g) \leq \Sigma^n_{i=1} w_i$ is said to be Pareto Optimal if there does not exist $(\bar{x}, \bar{g}) \in \mathbb{R}^n_+ \times \mathbb{R}_+$ satisfying $\Sigma^n_{i=1} \bar{x}_i + c(\bar{g}) \leq \Sigma^n_{i=1} w_i$ and $u_i(\bar{x}_i, \bar{g}) \geq u_i(x_i, g) \; \forall \; i=1,\ldots,n$ with at least one strict inequality. However, in one special case which is both very popular in the relevant literature as well as of considerable merit, the Pareto Optimality of voting equilibria is easy to establish.

Let $u_i(x_i, y) = x_i + v_i(y) \; \forall x_i, y) \in \mathbb{R}^2_+, \forall \; i=1,\ldots,n$, where $v_i : \mathbb{R}_+ \to \mathbb{R}$ is concave, strictly increasing and differentiable for $\forall$ $i=1,\ldots,n$. Let $c: \mathbb{R}_+ \to \mathbb{R}_+$ satisfy $c(0)=0$ and be convex, strictly increasing and differentiable. It is then well known that a necessary and sufficient condition for Pareto Optimality of $(x, y) \in \mathbb{R}^n_+ \times \mathbb{R}_+$ is that $\Sigma^n_{i=1} x_i + c(y) \leq \Sigma^n_{i=1} w_i$

$$\text{and } \Sigma^n_{i=1} \frac{dv_i(y)}{dy} = c'(y).$$

On the other hand if (t^*, y^*) is a voting equilibrium for some compromise function G with $\frac{\delta G(y^*)}{\delta y_i} > 0 \; \forall = 1,\ldots,n$, then it is necessary under the above assumptions on preferences and the cost function that $t^*_i c'(G(y^*)) = \frac{dv_i(Gy^*)}{dy} \; \forall \; i=1,\ldots,n.$

Since $\Sigma^n_{i=1} t^*_i = 1$, we get that $\Sigma^n_{i=1} \frac{dv_i(G(y^*))}{dy} = c'(G(y^*))$

i.e. $((w_i - t^*_i \; c(G(y^*)))^n_{i=1}, G(y^*))$ is a Pareto Optimal allocation. Incidentally, it is easy to verify that $(t^*, g(y^*))$ satisfies the following condition as well: $\forall$ $i=1,\ldots,n, G(y^*)$ solves

$$w_i - t^*_i c(g) + v_i(g) \rightarrow \max$$

$$\text{s.t. } g \geq 0,\ w_i - t^*_i c(g) \geq 0,$$

which is the defining property of a ratio equilibrium, due to Kaneko (1977).

Thus, in case of quasi-linear concave preferences and convex cost functions, a voting equilibrium where each person asks for a different outlay of the public good, is associated to an allocation from which society as a whole has no incentive to deviate. This may be inspite of the fact, that the compromise solution agrees with nobody's demands. This explains why different individuals ask for different amounts of the public good and yet are satisfied with a compromise solution. The compromise function determines what each individual is going to ask for and not necessarily what society will provide. This is particularly true in the quasi-linear case above where $G(y^*)$ is uniquely determined, where as y^* varies with the choice of G.

5. Conlusion :- In this paper we have essentially established the existence of a voting equilibrium in a simple mixed economy and the Pareto Optimality of the associated allocation when preferences are quasi-linear. The linearity of the compromise function, which was required in the proof of existence of a voting equilibrium, does not severely restrict our environment, since we still have available all functions which takes the weighted average of the individual bids as society's choice of public good consumption level. The assumptions on preferences and cost functions are natural ones in the literature.

The discussion of the efficiency of a voting equilibrium allocation with quasi-linear preferences, also implied that the role of the compromise function was largely to allow for the possibility of divergent claims and not necessarily in choosing a different allocation from the ones suggested by the conventional solution concepts. Compromise functions may thus expalin what has always been observed in a democratic society : Conflicting claims and yet unanimity in realized choices.

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