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THREAT BARGAINING GAMES WITH A VARIABLE POPULATION

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ABSTRACT

In this paper we establish links between desirable properties satisfied by familiar solutions to bargaining games with a variable population and the Nash equilibrium concept for threat bargaining games.

1. Introduction :- In the classical formulation of the bargaining problem, which was posed for the first time by Nash (1950), it is assumed that a fixed number of agents are involved. However, both in real situations as well as in theory, it is necessary to accommodate the possibility of a variable number of players. A series of papers by Lensberg (1987, 1986), Thomson (1982, 1983a, 1983b), Thomson and Lensberg (1983 and Lahiri (1988a) have dwelt on this problem of an adequate representation of bargaining games with a variable population. The conventional view of this problem focuses on the behaviour of agents in circumstances where new agents come in without their entry being accompanied by an expansion of opportunities. Lahiri (1988a) takes a view somewhat different from the conventional one. There the focus is on the behaviour of agents udder cirumstances where some agents breakaway from the bargaining process, leaving the rest to share the existing opportunities among themselves. It turns out that this approach is somewhat more relevant to the kind of analysis we visualize in this paper.

On yet another line of approach to the bargaining problem, there is a modest literature on variable threat games which for obvious reasons is referred to in this paper as threat bargaining games (See e.g. Owen (1982); Lahiri (1988b), Lahiri (1988c). Our objective in this paper is to analyse desirable properties of threat bargaining problems with a variable population.

In an n-person classical bargaining game the n players have to choose a payoff vector from a compact, convex set S of possible payoff vectors called the payoff space of the game. The choice must be by unanimous agreement of all n players. If they cannot reach unanimous agreement, then they obtain conflict payoffs $d_1, \ldots\ldots, d_n$. The payoff vector $d = (d_1, \ldots\ldots, d_n) \in S$ is called the threat point (or disagreement point) of the game.

There are two ways in which the above framework can be generalized. One is by considering the number of agents to be a variable. The other is to allow the threat point to be a variable. The two have been studied independently. The first mode of generalization allows for such concepts as stability of bargaining solutions and monotonicity of bargaining solutions with respect to changes in the number of agents. The second mode of generalization leads naturally to the domain of non-cooperative solution concepts once the choice of threat points is permitted to proceed without the premise of binding agreements. In this paper we propse a theory which allows a bargaining game to be generalized along both these directions simultaneously-hence the name threat bargaining games with a variable population.

2. <u>The Model</u>: We assume that there is an infinite population I of agents, indexed by the positive integers. Arbitrary finite subsets of I may be confronted by a problem. The family of these subsets is denoted $\mathcal{P}$. Given $p \in \mathcal{P}$, Σ^P is the class of problems that the group P may conceivably face. Each $(S,d) \in \Sigma^P$ is a pair where S is a subset of IR_+^P (The nonnegative portion of the $|P|$-dimensional Eucliden space with coordinates indexed by the members of p) and $d \in S$. Each point of S represents the von Neumann–Morgenstern utilities achievable by the members of p through some joint action. It is assumed that

1 S is a compact subset of IR_+^P containing at least one strictly positive vector;

2 S is convex;

3 S is comprehensive (i.e., if x, y IR_+^P, $x \in S$, and $x \geq y$, then $y \in S$).

From some characterizations of the bargaining problem we often require in addition

4 $\forall (S,d) \in \Sigma^P$, $\exists$ $x \in S$ such that $d > x$.

5. If $x, y \in S$ and $x \geq y$, then there exists $z \in S$ with $z > y$.

These assumptions are quite standard in the literature on bergaining games. Condition 5 is a non-level set assumption for the undominated boundary of S.

The set of all $|P|$- person bargaining games satisfying 1,2 and 3 is denoted Σ^P. The set of all P - person bargaining games satisfying 1,2,3 and 5 is denoted $\tilde{\Sigma}^P$. The set of all $|P|$- person bargaining games satisfying 1,2,3 and 4 is denoted $\bar{\Sigma}^P$. $\tilde{\Sigma}^P \cap \bar{\Sigma}^P$ is denoted Σ_*^P.

Finally, we set

$$\Sigma \equiv \bigcup_{P \in \mathcal{P}} \Sigma^P, \quad \tilde{\Sigma} \equiv \bigcup_{P \in \mathcal{P}} \tilde{\Sigma}^P, \quad \bar{\Sigma} \equiv \bigcup_{P \in \mathcal{P}} \bar{\Sigma}^P \text{ and } \Sigma_x \equiv \bigcup_{P \in \mathcal{P}} \Sigma^P$$

A solution is function $F : \Sigma \longrightarrow \bigcup_{P \in \mathcal{P}} \mathbb{R}^P$ and associating for each $P \in \mathcal{P}$ and to each $(S,d) \in \Sigma^P$, a unique point $F(S,d) \in S$ called the <u>solution outcome</u> of (S,d).

So far the threat point has been considered to be fixed. We now allow the threat point to be a variable and then look at the possible implications of such a generalization. For the sake of simplicity we assume a scenario as follows:

Consider a bargaining situation amongst agents indexed by the set P. Confronted with a set S of feasible alternatives, each agent must choose a disagreement outcome d_i, $i \in P$, such that $d = (d_i)_{i \in P}$ belongs to a prespecified set $D(S) \subseteq S$. We could have allowed for strategies resulting in conflict outcomes; however for the present purpose there is no loss of generality in assuming that each agent directly announces his disagreement outcome without resorting to the device of communicating in some other language. To make the analysis consistent with the above framework, we assume that

6. $\forall P \in \mathcal{P}$, $\forall S \subseteq \mathbb{R}^P_+$ where S is compact, convex, comprehensive and has at least one strictly positive outcome, $D(S) \subseteq \{x \in S / \exists\ y \in S \text{ with } y > x\}$.

A pair $(S, D(S))$ satisfying Condition 6 is called a $|P|$- person threat bargaining game indexed by member of P.

Let Ω^P be the set of all $|P|$ - person threat bargaining games indexed by members of P.

$$\text{Let } \tilde{\Omega}^P = \{(S, D(S)) \in \Omega^P / \quad S \text{ Satisfies } (5)\}$$
$$\bar{\Omega}^P = \{(S, D(S)) \in \Omega^P / \quad S \text{ satisfies } (4)\}$$
$$\Omega_*^P = \tilde{\Omega}^P \cap \bar{\Omega}^P .$$

We define,

$$\Omega = \bigcup_{P \in \mathcal{P}} \Omega^P, \quad \tilde{\Omega} = \bigcup_{P \in \mathcal{P}} \tilde{\Omega}^P, \quad \bar{\Omega} = \bigcup_{P \in \mathcal{P}} \bar{\Omega}^P, \quad \Omega_* = \bigcup_{P \in \mathcal{P}} \Omega_*^P$$

A $|P|$ - person threat bargaining game $(S, D(S)) \in \Omega^P$ equipped with a solution $F: \sum \to \bigcup_{P \in \mathcal{P}} \mathbb{R}^P$ is an ordered triplot $\langle S, D(S), F \rangle$.

An <u>equilibrium threat strategy</u> for $\langle S, D(S), F \rangle$ is a point $d^* \in D(S)$ such the $\forall$ $i \in P$,

$$F_i(S, d^*) \geq F_i(S, d^*_{-i} | d_i) \quad \forall (d^*_{-i} | d_i) \in D(S) \text{ where } (S, D(S)) \in \Omega^P .$$

Here $(d^*_{-i} | d_i)$ is the element of $D(S)$ whose j^{th} coordinate for $j \neq i$, $j \in P$ is d^*_j and whose i^{th} coordinate is d_i.

Our definition of an <u>equilibrium threat strategy</u> corresponds to the Nash equilibrium of the resulting non-cooperative game. In sum we require the equilibrium threat strategy to be self-enforceable in the sense that no agent stands to gain by unilaterally deviating from his component of the equilibrium threat strategy.

We conclude this section by introducing some additional notation. Given P, Q in $\mathcal{P}$ with $P \subset Q$, y a point of IR^Q, and T, a subset of IR^Q, y_p and T_p designate the projections on IR^p of y and T, respectively.

Given $(S, D(S)) \in \Omega^Q$, and $\bar{d} \in D(S)$, we shall be primarily interested in the following sets:

$$T = \left\{ x \in S / x_{Q-p} = \bar{d}_{Q-p} \right\}_p .$$

$$D(T) = \left\{ d \in D(S) / d_{Q \setminus p} = \bar{d}_{Q \setminus p} \right\} p.$$

where $P \subset Q$.

The following result is immediate :

Theorem 1 :- (a) If $(S, D(S)) \in \Omega^Q$ then $(T, D(T))$ as defined above belongs to Ω^P

(b) If $(S, D(S)) \in \tilde{\Omega}^Q$ then $(T, D(T)) \in \tilde{\Omega}^P$

(c) If $(S, D(S)) \in \bar{\Omega}^Q$ then $(T, D(T)) \in \bar{\Omega}^P$

(d) If $(S, D(S)) \in \Omega^Q_*$ then $(T, D(T)) \in \Omega^P_*$

Proof : It is easy to verify the results by noting that the respective inequalities are satisfied.

3. <u>Stable Equilibrium Threat Strategies</u> : The question we would like to answer in this paper is about what happens when a group of bargainers break away from bargaining. Clearly the set of feasible alternatives reduce in dimension. But, does the equilibrium threat strategy for the remaining bargainers remain the same as in the original game? Consider a situation where an impartial arbitrator decides on the final outcome given the feasible set of alternatives once the players announce their disagreement payoff. In this situation, if the equilibrium threat strategies remain the same as before for the bargainers who remain, any further disclosure of information on the part of the agents will be prevented. A second question that arises is whether after some players leave, do the final outcome for the remaining players improve given that in the original as well as in the new situation the players announce their equilibrium threat strategies. A third question may be posed in a somewhat different context. Suppose in a bargaining game, a certain subset of the players are given their solution payoffs resulting from equilibrium threat strategies and then the remaining players start bargaining among themselves. Do the equilibrium threat strategies of the remaining players remain the same? A host of such questions may be posed in this framework, but for the present let us formalize the conepts we have outlined above.

An equilibrium threat strategy d^* for $\langle S, D(S), F\rangle \in \Omega^Q$ is said to be <u>strategically stable</u> if $\forall\, P \subset Q$, d^*_P is an equilibrium threat strategy for $\langle T, D(T), F\rangle$ where

$$T = \left\{x \in S / \ x_{Q \setminus P} = d^*_{Q \setminus P}\right\}_P, \quad D(T) = \left\{d \in D(S)/d_{Q \setminus P} = d^*_{Q \setminus P}\right\}_P.$$

An equilibrium threat strategy d^* for $\langle S, D(S), F\rangle \in \Omega^Q$ is said to be <u>strategically monotone with respect to changes in the number of agents</u> if d^* is strategically stable for $\langle S, D(S), F\rangle \in \Omega^Q$ and $\forall\, P \subset Q$, $F(T, d^*_P) \geq F_P(S, d^*)$ where $T = \left\{x \in S / x_{Q \setminus P} = d^*_{Q \setminus P}\right\}_P$.

An equilibrium threat strategy d^* for $\langle S, D(S), F\rangle \in \Omega^Q$ is said to be <u>strategically constant</u> if $\forall\ P \subset Q$, d^*_P is an equilibrium threat strategy for $\langle U, D(U), F\rangle$ where

$$U = \left\{x \in S/x_{Q \setminus P} = F_{Q \setminus P}(S, d^*)\right\}_P$$

and $D(U) = \left\{d \in D(S)/d_{Q \setminus P} = d^*_{Q \setminus P}\right\}_P$.

In this section we have introduced a battery of new concepts. We shall now briefly interpret these concepts.
Strategic stability means that if some agents break-away from the bargaining game, then the original equilibrium threat strategies of the surviving players remain as equilibrium threat strategies in the modified bargaining game. An equilibrium threat strategy is said to be strategically monotone with respect to changes in the number of agents if it is strategically stable and further the payoffs to the surviving players in the modified game is at least as much as the payoffs accruing to them in the original game. Strategic constancy means that if some agents break away

from the bargaining game with their arbitrated outcome resulting from an equilibrium threat strategy ensemble, then the components of the equilibrium threat strategy ensemble corresponding to the surviving members will still be their equilibrium threat strategy.

Stability Properties of Solutions satisfying monotonicity with respect to the disagreement point :

In this section we first formulate a definition of monotonicity with respect to the disagreement point and then show that solution satisfying this property yield equilibrium threat strategies which are strategically stable.

Monotonicity With Respect To The Disagreement Point (MON-d): For all $P \in \mathcal{P}$ and $\forall (S,d) \in \sum^P$, $(S',d') \in \sum^P$, for all $i \in P$, if $S' = S$, $d'_i > d_i$ and $d'_j = d_j$ for $j \neq i$, $j \in P$, then $F_i(S',d') > F_i(S,d)$.

A variation of this property has been applied for the resolution of two person threat bargaining games in Lahiri (1988b, 1988c). Since the property is self explanatory we shall content ourselves by stating that many familiar bargaining solutions satisfy this property on suitable feasible sets.

Theorem 2 : Let $F : \sum \rightarrow \bigcup_{P \in \mathcal{P}} \mathbb{R}^P$ satisfy monotonicity with respect to the disagreement point. Let $Q \in \mathcal{P}$, $(S,D(s)) \in \Omega^Q$ and let d^* be an equilibrium threat strategy for $(S,D(S))$. Then d^* is strategically stable.

<u>Proof</u>: Let $(S,D(S)) \in \Omega^Q$ and let d^* be an equilibrium threat strategy for $(S,D(S))$. Let $P \subset Q$ and suppose towards a contradiction that d^*_p is not an equilibrium threat strategy for $(T,D(T))$ where

$$T = \left\{ x \in S / x_{Q \sim p} = d^*_{Q \sim p} \right\}_p$$

$$D(T) = \left\{ d \in D(S)/d_{Q \sim p} = d^*_{Q \sim p} \right\}_p$$

Hence there exists $i \in P$ and $((d^*_p)_{-i}/d_i) \in D(T)$ such that

$$F_i(T,((d^*_p)_{-i} \mid d_i)) > F(T, d^*_p)$$

where $d_i \neq d_i^*$.

By monotonicity with respect to the disagreement point, $d_i > d^*_i$.

Consider the strategy $(d^*_{-i} \mid d_i) \in D(S)$. (By the definition of $D(T)$, this condition is satisfied).

Once again, by monotonicity with respect to the disagreement point,

$$F_i(S,(d^*_{-i} \mid d_i)) > F_i(S,d^*).$$

This contradicts that d^* is an equilibrium threat strategy for $(S,D(S))$ and proves the theorem.

Q.E.D.

It is worth noting that the Kalai-Smorodinsky (1975) solution and the Kalai (1977) solution satisfies (MON-d) on $\sum$. The Nash (1950) solution satisfies (MON-d) on the subset of consisting of games with smooth Pareto-optimal individually rational bounderies.

<u>Strategic Monotonicity With Respect To Changes In The Number of Agents:</u>

So far we have established conditions which guarantee that the equilibrium threat strategies of a bargaining game are strategically stable. We shall now try to find conditions which guarantee that equilibrium threat strategies satisfy monotonicity with respect to changes in the number of agents. One obvious condition which in conjunction with monotonicity with respect to the disagreement point leads to the desired result is the following:

<u>Restricted Monotonicity With Respect to changes in the number of agents</u> (R.MON):- For all $P, Q \in \mathcal{P}$ with $P \subset Q$, for all $(S,d') \in \sum^{P}$, $(T,d) \in \sum^{Q}$, if $S = \{x \in T / x_{Q \setminus P} = d_{Q \setminus P}\}_P$, $d' = d_p$, then $F(S,d') \geqq F_p(T,d)$.

<u>Theorem 3</u> : Let $F : \sum \rightarrow \bigcup_{P \in \mathcal{P}} IR^{P}$ satisfy (MON-d) and (R.MON). Then if d^* is an equilibrium threat strategy for $(S,D(S)) \in \Omega^{Q}$, d^* is strategically monotone with respect to changes in the number of agents.

<u>Proof:</u> Obvious.

In Lahiri (1988a) it is shown that a class of solutions defined with respect to a reference function on the set of bargaining games $\sum_*$ satisfies (R.MON) under certain additional conditions. This class includes well known solutions like those of Kalai-Smorodinsky (1975).

<u>Strategic Constancy and Bargaining Games</u> :-

The third criteria we have in mind is the strategic constancy of equilibrium threat strategies. Our objective in this section is to show that this property is satisfied by solutions which obey a multilateral stability property enunciated by Lensberg (1987, 1988).

<u>Multilateral Stability of Bargaining Solutions</u> (M. STAB):- For all $P, Q \in \mathcal{P}$ with $P \subset Q$, for all $(T,d) \in \sum^Q$, $(S,d') \in \sum^P$ with $d' = d_P$ and

$$S = \left\{ x \in T / x_{Q \setminus P} = F_{Q \setminus P}(T,d) \right\}_P$$

$$F(S,d') = F_P(T,d)$$

Our next theorem proves that if F is a solution satisfying M. STAB and if d* is an equilibrium threat strategy, then d* satisfies strategic constancy.

<u>Theorem 4</u> :- Let $F : \sum \rightarrow \bigcup_{P \in \mathcal{P}} \mathbb{R}^P$ satisfy (M.STAB) and let d* be an equilibrium threat strategy for $(T, D(T)) \in \Omega^Q$. Then d* is strategically constant.

<u>Proof</u>: Suppose otherwise. Then there exists $P \subset Q$,

$$S = \left\{ x \in T / \; x_{Q \setminus P} = F_{Q \setminus P}(T, d^*) \right\}_P$$

$$D(S) = \left\{ d \in D(T) / d_{Q \setminus P} = d^*_{Q \setminus P} \right\}_P, \quad \text{and}$$

$$((d^*_P)_{-i} | d_i) \in D(S) \text{ for some } i \in P \text{ with}$$

$$F_i(S, ((d^*_P)_{-i} | \; d_i)) > F_i(S, d^*_P).$$

Consider the strategy $(d^*_{-i} | d_i) \in D(T)$. It does so by the definition of $D(S)$. Since d^* is an equilibrium threat strategy,

$$F_i(T, (d^*_{-i} | d_i)) \leq F_i(T, d^*).$$

But F satisfies M. STAB. Hence,

$$F_i(T, (d^*_{-i} | d_i)) = F_i(S, (d^*_{-i} | d_i)_p)$$

$$F_i(T, d^*) = F_i(S, d^*_p).$$

However, $(d^*_{-i} | d_i)_p = ((d^*_p)_{-i} | d_i)$.

$$\therefore F_i(T, d) \geq F_i(T, (d^*_{-i} | d_i)) = F_i(S, ((d^*_p)_{-i} | d_i)) > F_i(S, d^*_p)$$

which contradicts what we have just obtained above.

Hence d^* is strategically constant.

Q.E.D.

Lensberg (1987, 1988) shows that the Nash (1950) solution satisfies M.STAB on Σ . Hence equilibrium threat strategies with respect to the Nash bargaining game satisfies strategic constancy.

<u>Conclusion</u>: In this paper we have established links between desirable properties satisifed by familiar solutions to bargaining games with a variable population and the Nash equilibrium concept for threat bargaining games. The concept of a threat bargaining game with a variable population has been developed, and welcome features of the resulting equilibrium threat strategies have been shown to follow from the conventional properties required of threat bargaining games. This helps to establish a connection between the literature on information dissemination in bargaining and the literature on allocation of a fixed supply of resources among a variable population.

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