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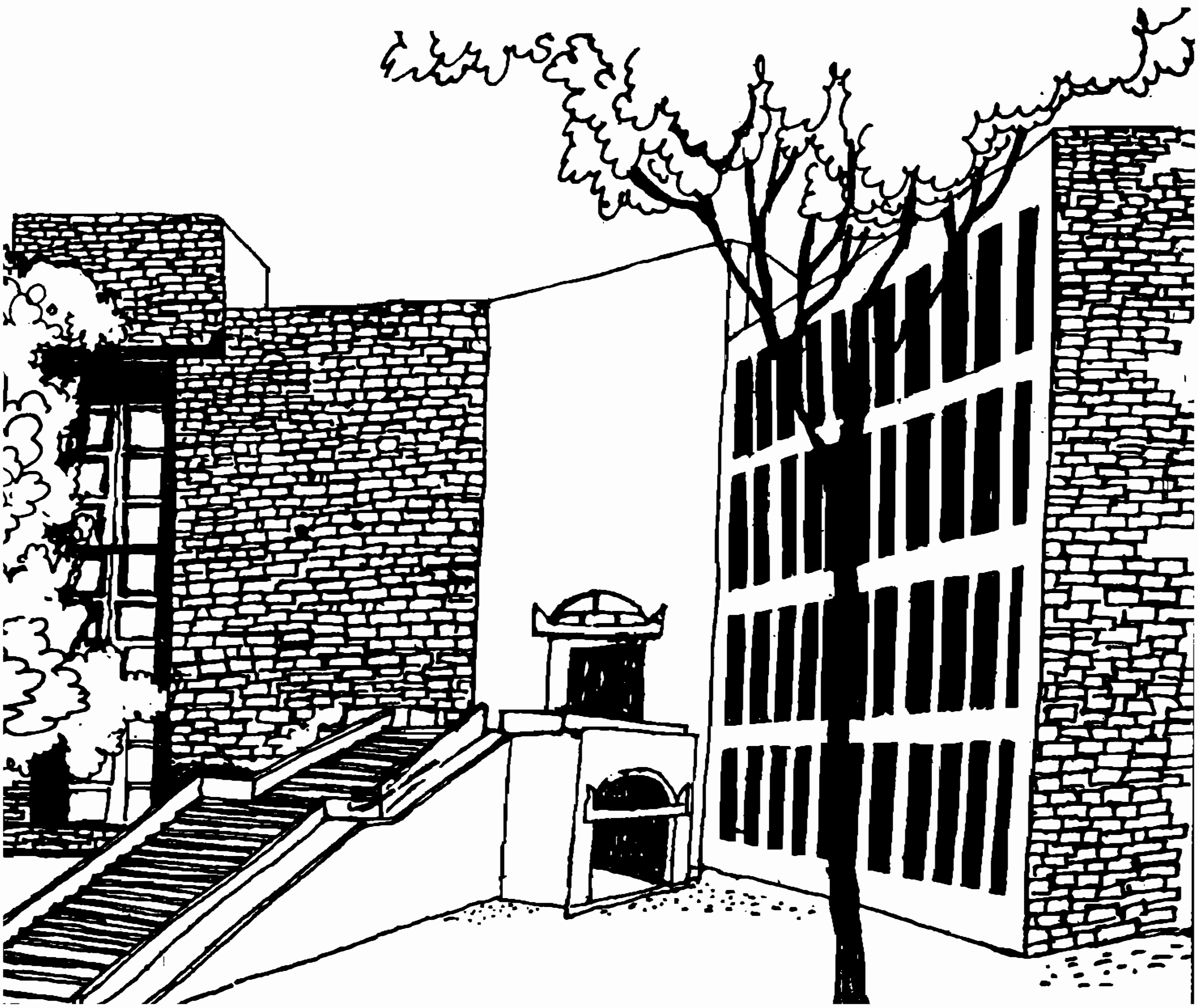

A SIMPLE PROOF PROVIDING AN AXIOMATIC CHARACTERIZATION OF THE KALAI-SMORODINSKY SOLUTION

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ABSTRACT

In this paper we provide a simple proof for the axiomatic characterization of the Kalai-Smorodinsky Solution

1. Introduction: In their book entitled *Axiomatic Theory of Bargaining with a Variable Number of Agents,* Thomson and Lensberg (1989) make the following assertion on page 20 (Chapter 2):

> *Although the extension of the definition of the Kalai-Smorodinsky solution to the n-person case itself cases no problem, the generalization of the preceding results to the n-person case is not as straightforward as was the case of the extensions of the results concerning the Nash solution from n = 2 to arbitrary n. First of all, for n > 2, the n-person Kalai-Smorodinsky solution satisfies w.p.o only.*

Except for the last sentence of the above quote, the rest is a litle difficult to accept, since a slight variation of their own proof of the axiomatic characterization of the egalitarian solution (i.e. Theorem 2.5, appearing on page 22 of the same book) yields the desired axiomatic characterization of the Kalai-Smorodinsky solution. As it turns out, the proof is extremely simple and merits reporting. That is what we do, in the rest of the paper.

2. The Model: An $n(\varepsilon \mathbf{N})$ - person bargaining problem is a nonempty subset S of $\mathbb{R}^n_+$ (: the non-negative orthant n-dimensional Euclidean space) satisfying the following properties:

i) S is a convex, compact subset of $\mathbb{R}^n_+$ containing at least one strictly positive vector.

ii) For all $x, y \varepsilon \mathbb{R}^n_{++}$, if $x \varepsilon S$ and $x \geq y$, then $y \varepsilon S$.

Let B^n be the family of all n-person bargaining problems.

A solution on B^n is a function $F: B^n \rightarrow \mathbb{R}^n_+$ such that $F(S) \varepsilon S$ for all $S \varepsilon B^n$.

Let $S \varepsilon B^n$. The **utopia point** of S is the vector $u(S) \varepsilon \mathbb{R}^n_{++}$ whose i^{th} co-ordinate $u_i(S)$ is defined as follows:

$u_i(S) = \max \{x_i / x \varepsilon S\}, i = 1, ..., n$

Clearly $u_i(S) > o, \forall i = 1, ..., n$

The Kalai-Smorodinsky solution $K: B^n \rightarrow \mathbb{R}^n_+$ is defined as follows: given $S \varepsilon B^n$, $K(S) = \bar{t}(S) u(S)$, where $\bar{t}(S) = \max \{t \geq o / t u(S) \varepsilon S\}$.

3. The Axioms: Let $F : B^n \rightarrow \mathbb{R}^n_+$ be a solution.It is said to satisfy

Weak Pareto Optimality (WPO), if $\forall S \varepsilon B^n$, $x \varepsilon \mathbb{R}^n_{++}$, $x_i > F_i(S) \forall i = 1, ..., n, \Rightarrow x \notin S$.

Symmetry (SY) if $\forall S \varepsilon B^n$ and all $\pi \varepsilon \Pi^n$, $\pi(S) = S$ implies $F_i(S) = F_j(S) \forall i, j = 1, .., n$

(Here Π^n is the set of permutations of order n. Given $\pi \varepsilon \Pi^n$ and $x \varepsilon \mathbb{R}^n$, let $\pi(x) = (x_{\pi(1)}, ..., x_{\pi(n)})$. Also, given $S \subseteq \mathbb{R}^n$, let $\pi(S) = \{\pi(x) / x \varepsilon S\}$.)

Scale Invariance (S.INV) if $\forall S \varepsilon B^n$ and all $l \varepsilon L$, $F(l(S)) = l(F(S))$.

Here L is the class of positive, independent person-by-person and linear transformations on $\mathbb{R}^n$. Given $S \subseteq \mathbb{R}^n$, $l(S) = \{l(x)/ x \varepsilon S\}$.

Individual Monotonicity (I.MON) if for all $S, T \varepsilon B^n$, for all i, if $u_j(S) = u_j(T)$ for $j \neq i$ and $S \subseteq T$, then $F_i(T) \geq F_i(S)$.

Restricted Monotonicity (R.MON) if for all $S, T \varepsilon B^n$, if $u(S) = u(T)$ and $S \subseteq T$, the $F(T) \geq F(S)$.

Clearly (I.MON) $\rightarrow$ (R.MON).

4. The Main Theorem:

Theorem: The only solution on B^n to satisfy WPO, SY, S.INV and LMON is K.

Proof: That K satisfies the above properties is obvious. Thus assume that F is a solution on B^n satisfying the above properties and $S \in B^n$. By S.INV assume $u(S) = e$ i.e. $u_i(S) = 1 \; \forall \; i = 1, \ldots, n$. Thus there exists $b > 0$ such that $K(S) = be$.

Let T = convex comprehensive hull $\{b, e_1, e_2, \ldots, e_n\}$ where e_i is the i^{th} unit co-ordinate vector. T is symmetric. Hence by WPO and SY, $F(T) = be$

By R.MON, since $u(T) = u(S)$ and $T \subseteq S$, $K(S) = be = F(T) < F(S)$.

Case 1: $K(S) \varepsilon P(S) = \{ x \varepsilon S / y \geq x, y \varepsilon S \Rightarrow y = x \}$.

Then $K(S) = F(S)$.

Case 2: $K(S) \notin P(S)$.

Thus there exists $x \varepsilon S$ with $x_i \geq K_i(S) \; \forall \; i$ and $x_j > K_j(S)$ for some j.

Since $1 = u_j(S) \geq x_j$, we get $1 > K_j(S) = b$.

Thus $u_i(S) > b \; \forall \; i = 1, \ldots, n$

Thus for $\upsilon \varepsilon \mathbf{N}$, υ large $u_i(S) > b + 1/\upsilon$.

Let T^υ = convex hull $\{S, (b + 1/\upsilon) e\}$

$\therefore \quad K(T^\upsilon) = (b + 1/\upsilon) e$

Since $(b + 1/\upsilon) e \varepsilon P(T^\upsilon)$, by case 1, $F(T^\upsilon) = (b + 1/\upsilon) e$

But $u(T^\upsilon) = e$ and $S \subseteq T^\upsilon \; \forall \; \upsilon$ (large) $\Rightarrow F(S) \leq F(T^\upsilon) = (b + 1/\upsilon) e \; \forall \; \upsilon$ (large)

Taking $\upsilon \rightarrow \infty$, we get $F(S) = be = K(S)$. Q.E.D.

Reference:

1. W.Thomson and T Lensberg (1989): *Axiomatic Models of Bargaining with a Variable Number of Agents*, Cambridge University Press.