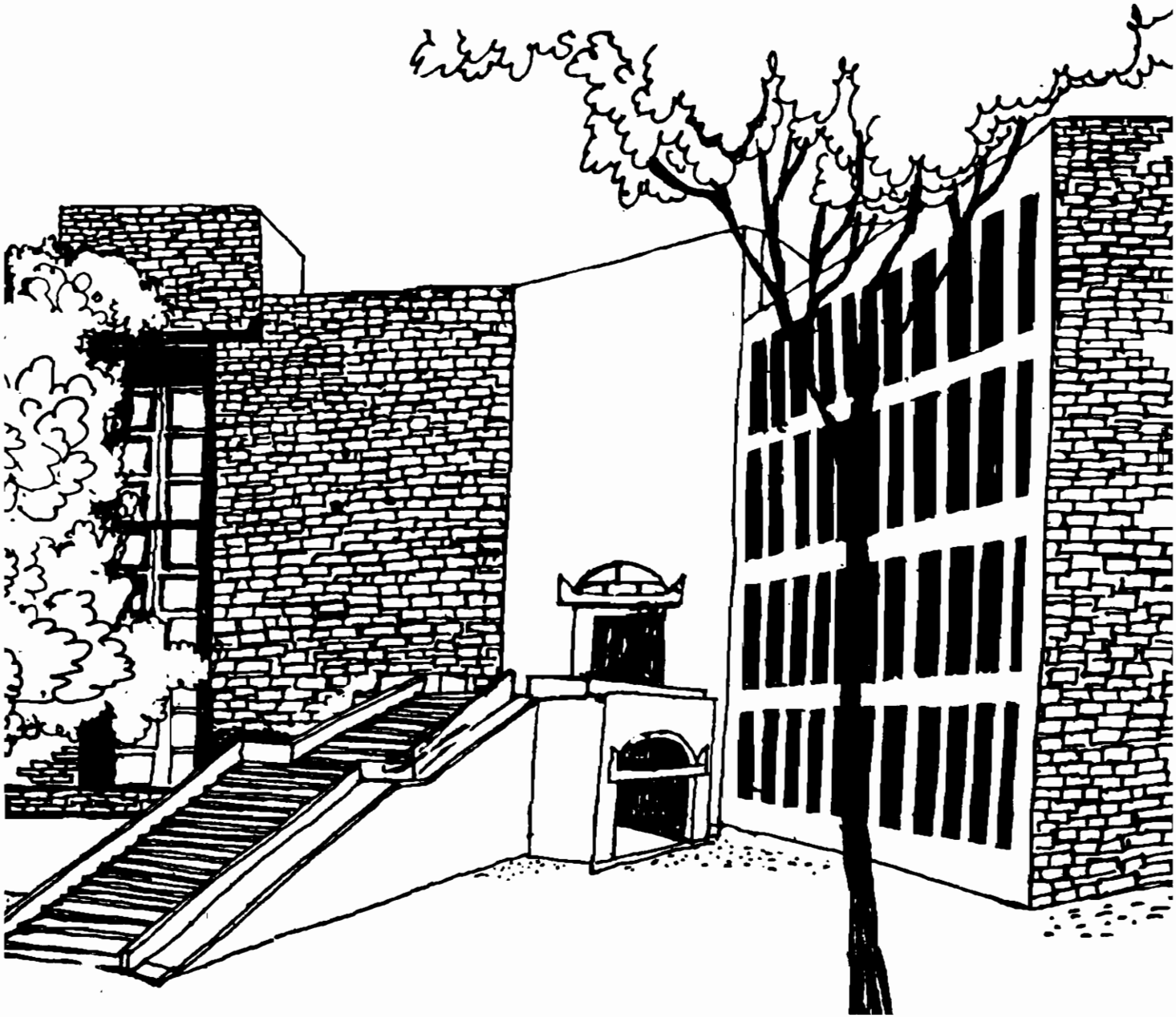
विद्यादिनियोगाद्विकासः

IIM

AHMEDABAD

Working Paper

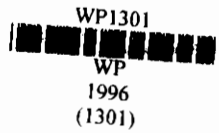


A RECONSIDERATION OF THE ADDITIVE CHOICE FUNCTION

By

Somdeb Lahiri

W P No.1301
March 1996



The main objective of the working paper series of the IIMA is to help faculty members to test out their research findings at the pre-publication stage.

INDIAN INSTITUTE OF MANAGEMENT
AHMEDABAD - 380 015
INDIA

Abstract

The purpose of this paper is to provide axiomatic characterizations of the additive and weighted additive choice functions which are now defined over the entire domain of convex, compact, comprehensive choice problems.

PURCHASED
APPROVAL
GRATIS/EXCHANGE
PRICE
ACC NO.
VIKRAM SARABHAI LIBRARY
I. I. M. AHMEDABAD

A Reconsideration of the Additive Choice Function

1 Introduction

Choice theory which dawned with the seminal paper of Nash written in 1950, has by now developed into a well defined body of mathematics, concerned with choosing a point from a compact, convex, comprehensive feasible subset of the non-negative orthant of a finite dimensional Euclidean space, each such feasible set admitting a strictly positive vector. Axiomatic choice theory is concerned with the axiomatic characterization of rules which assign an alternative to each such choice problem in a given family of choice problems. We shall here be concerned with two dimensional choice problems.

Following the choice function suggested by Nash, the other well known choice functions are the relative egalitarian due to Kalai and Smorodinsky [1975], egalitarian due to Kalai [1977], lexicographic egalitarian due to Chun and Peters [1988], equal loss due to Chun [1988], lexicographic equal loss due to Chun and Peters [1991] and the equal area due to Anbarci and Bigelow [1994]. Some of the other choice functions have been studied on more relevant domains in Lahiri [1996]. However, the simplest of all solutions i.e., the one which maximizes the sum of the coordinates from amongst all feasible vectors has been a rather mute spectator of a spectacular pageantry in which all these other choice functions participate. Except for a significant axiomatic characterization by Myerson [1981], very little attention has been devoted to this choice function: the utilitarian choice function. The reason is that this choice function (as a single valued mapping) is not well defined for a very large class of meaningful and non pathological choice problems. The purpose of this paper is to suggest a way out of this difficulty, so that much of applied research which uses maximization of the sum of the coordinates of vectors in a feasible set of vectors will now have a theoretical underpinning. We also suggest a variant of a choice function due to Cao [1981], which is also well defined on the larger domain and yet satisfies scale translation covariance property. Some remarks about related results due to Peters [1986] are given, to put earlier results in proper perspective. In an appendix to this paper we prove a variant of a result in Peters [1986], which is valid on our domain.

2 The Model

We consider two dimensional choice problems only. A (two dimensional) choice problem is a non-empty subset S of $\mathbb{R}^2_+$ (: the non-negative quadrant of two dimensional Euclidean space), satisfying the following properties:

i) S is compact (: closed and bounded), convex

ii) S is comprehensive i.e. $0 \leq y \leq x \epsilon S \rightarrow y \epsilon S$

iii) there exists $x \epsilon S$ such that $x \gg 0$ (i.e. if $x = (x_1, x_2)$ then $x_1 > 0$, $x_2 > 0$). Let Σ^2 be the class of all choice problems.

A choice function (or solution) is a function $F : \Sigma^2 \rightarrow \mathbb{R}^2_+$ such that $F(S) \epsilon S \ \forall \ S \epsilon \Sigma^2$.

Given $S \epsilon \Sigma^2$, let $u(S) \equiv \left\{x \epsilon S / x_1 + x_2 \geq y_1 + y_2 \ \forall \ y = (y_1, y_2) \ \epsilon S\right\}$. u(S) is non-empty for all $S \epsilon \Sigma^2$. Further u(S) is a compact convex subset of

$\Delta_c \equiv \left\{x \epsilon \mathbb{R}^2_+ / x = (x_1, x_2),\ x_1 + x_2 = c\right\} \ \forall \ S \epsilon \Sigma^2$ for some c>0. However, u(S) is in general not a singleton.

Example: Let $S = \left\{x \epsilon \mathbb{R}^2_+ / x = (x_1, x_2),\ x_1 + x_2 \leq 1\right\}$. Then $u(S) = \Delta_1$.

Let $a_1(S) \equiv \max \left\{x_1 / \exists\ x_2 \geq 0 \ with \ (x_1, x_2) \ \epsilon u(S)\right\}$,

$b_1(S) \equiv \min \left\{x_1 / \exists\ x_2 \geq 0 \ \ with \ \ (x_1, x_2) \ \epsilon \ u(S)\right\}$.

Let $a(S) = \left(a_1(S),\ a_2(S)\right)$,

$b(S) = \left(b_1(S),\ b_2(S)\right) \ \epsilon \ U(S)$

Clearly, a(S) and b(S) are well defined for all $S \epsilon \Sigma^2$ and $u(S) = \left\{t a(S) + (1-t)\ b(S) / t \epsilon [0,1]\right\}$.

We define the additive choice function $\bar{A} : \Sigma^2 \rightarrow \mathbb{R}^2_+$ as follows:

$$\bar{A}(S) = \frac{1}{2}\left(a(S) + b(S)\right) \ \forall \ S \epsilon \Sigma^2.$$

We are basically interested in the axiomatic characterization of this choice function, which is nothing but the expected value of the random vector which has a uniform distribution on u(S).

3 Some Axioms

Let $F : \Sigma^2 \rightarrow \mathbb{R}^2_+$ be a choice function.

1) Weak Parcto Optimality (WPO):

$\forall\, S \in \Sigma^2,\ F(S) \in W(S)$, where $W(S) = \{x \in S / y \succ x \rightarrow y \notin S\} \forall\, S \in \Sigma^2$.

2) Parcto Optimality (PO):

$\forall\, S \in \Sigma^2,\ F(S) \in P(S)$, where

$P(S) = \{x \in S / y \geq x,\ y \in S \rightarrow y = x\} \forall\, S \in \Sigma^2$.

3) Scale Translation Covariance (STC):

$\forall\, S \in \Sigma^2,\ \forall\, c \in \mathbb{R}^2_+,$ *if* $c = (c_1, c_2)$ then

$F(cS) = \{c_1 F_1(S),\ c_2 F_2(S)\}$, given that

$cS = \{(c_1 x_1,\ c_2 x_2) / (x_1, x_2) \in S\}$.

4) Homogeneity (HOM):

$\forall\, S \in \Sigma^2,\ \forall\, t > 0,\ F(tS) = tF(S)$, when

$tx = (tx_1, tx_2)\ \forall\, x = (x_1, x_2) \in R^2_+$ and tS = {tx/x ε S}.

5) Additivity (Addi):

$\forall\, S \in \Sigma^2,\ T \in \Sigma^2,\ F(S + T) = F(S) + F(T)$.

6) Super Additivity (S Addi):

$\forall\, S, T \in \Sigma^2,\ F(S+T) \geq F(S) + F(T)$.

7) Partial Super Additivity (PS Addi):

$\forall\, S, T \in \Sigma^2,\ F(S+T) \geq F(S)$.

8) Nash's Independence of Irrelevant Alternatives (NIIA):

$\forall\, S, T \in \Sigma^2,\ S \subset T,\ F(T) \in S \rightarrow F(S) = F(T)$.

9) Translation Covariance (TC):

$\forall\, S \in \Sigma^2,\ c \in \mathbb{R}^2$ *if* $S(c) = \{y \in \mathbb{R}^2 / y \leq x + c,\ x \in S\}$,
then $F\{S(c)\} = F(S) + c$.

10) Symmetry (SYM):

$\forall\, S \in \Sigma^2$ *such that* $(x_1, x_2) \in S \leftrightarrow (x_2, x_1) \in S,\ F_1(S) = F_2(S)$.

11) Convex Linearity (C. Lin):

$\forall\, S,\ T \in \Sigma^2,\ F\{\alpha S + (1-\alpha)T\} = \alpha F(S) + (1-\alpha)F(T)$ *if* $\alpha \in [0,1]$.

12) Binary Additivity (B. Addi):

$\forall\, S,\ T \in \Sigma^2$ *with* $U(S) = \{\bar{A}(S)\}$ and $U(T) = \{\bar{A}(T)\}$ if

V = comprehensive convex hull {S, T}, then

$$F(V) = \frac{1}{2}\left[F(S) + F(T)\right] \quad \text{if} \quad F_1(S) + F_2(S) = F_1(T) + F_2(T).$$

Let us first mention that $\bar{A}$ does not satisfy STC and NIIA.

Example:

Let $T = \left\{x \in \mathbb{R}^2_+ / (x_1, x_2) = x,\ x_1 + x_2 \leq 1\right\}$,

$$S = \textit{Convex hull}\left\{(0,0),\ (0,1),\ \left(\frac{1}{2},\frac{1}{2}\right),\ \left(\frac{1}{2},0\right)\right\}.$$

Clearly $S \subset T$ and $\bar{A}(T) = \left(\frac{1}{2},\ \frac{1}{2}\right) \in S$. However $\bar{A}(S) = \left(\frac{1}{4},\ \frac{3}{4}\right)$. Thus $\bar{A}$ does not satisfy NIIA.

We will however, modify $\bar{A}$ somewhat later to take care of STC. Observe that:

i) PO → WPO

ii) STC → HOM

iii) Addi → S Addi → PS Addi

iv) Addi + HOM → C. LIN

4 A Result on the Additive Choice Function

Theorem 1:

The only choice function on Σ^2 to satisfy PO, SYM, C.LIN and B. Addi is $\bar{A}$.

Proof:

The proof that if F satisfies PO, SYM and C.LIN, then $F(S) \in \underset{x \in S}{argmax}\left[x_1 + x_2\right]\ \forall\ S \in \Sigma^2$ is the relevant portion of the proof of theorem 1 in Myerson [1981]. If in addition F satisfies B.Addi the following argument holds:

Let $V \in \Sigma^2$ and let $h_i(V) = \max\{x_i / x \in V\}$, $i = 1,2$. Suppose $\{\bar{A}(V)\}$ is a strict subset of U(V). (If $u(V) = \{\bar{A}(V)\}$, there is nothing more to be proved).

Case 1: $a(V) \in \mathbf{R}^2_+ \setminus \mathbf{R}^2_{++}$, $b(V) \in \mathbf{R}^2_+ \setminus \mathbf{R}^2_{++}$.

In this case $V = \Delta_c$ for some $C > 0$. By WPO and SYM, $F(V) = \bar{A}(V)$.

Case 2: $a(V) \in \mathbf{R}_{++}$, $b(V) \in \mathbf{R}^2_{++}$.

Let S = Convex comprehensive hull $\{(0, h_2(V)), \{x \in V / x_2 \leq a_2(V)\}\}$.

T = Convex comprehensive hull $\{(h_1(V), 0), \{x \in V / x_1 \leq b_1(V)\}\}$.

Clearly V = Convex comprehensive hull {S,T}

Further, $u(S) = \{\bar{A}(S)\} = \{a(V)\}$, $u(T) = \{\bar{A}(T)\} = \{b(V)\}$.

Thus F(S) = a(V), F(T) = b(V).

By *B.Addi*, $F(V) = \bar{A}(V)$.

Case 3: $a(V) \in \mathbf{R}^2_+ \setminus \mathbf{R}^2_{++}$, $b(V) \in \mathbf{R}^2_{++}$.

In this case let T be as in Case 2 and let

$S = \{x \in V / x_2 \leq a_2(V)\}$

Once again V = Comprehensive convex hull {S,T} and from here on the argument is as in Case 2.

Case 4: $a(V) \in \mathbf{R}^2_{++}$, $b(V) \in \mathbf{R}^2_+ \setminus \mathbf{R}^2_{++}$

In this case let T be as in Case 2 and let $S = \{x \in V / x_2 \leq (V)\}$

Again V = Comprehensive convex hull {S,T} and the resulting argument is as in Case 2.

Thus $F(V) = \bar{A}(V)$ in all cases Q.E.D.

Remarks:

1) The theorem due to Myerson [1981] which we refer to in our proof is valid only on a subdomain of Σ^2 for which $u(S) = \{\bar{A}(S)\}$. However, the same proof works for us.

2) We have shown that $\bar{A}$ satisfies PO, SYM, HOM and Addi. Thus $\bar{A}$ satisfies PO, SYM, HOM, and PS. Addi. Peters [1986] contains a theorem to the effect that the egalitarian solution due to Kalai [1977], is the only solution to satisfy WPO, SYM, HOM and PS. Addi. However, his domain is a nonconventional one and is different from ours. On our domain the egalitarian solution satisfies WPO, SYM, HOM and PS. Addi. as well. Thus a uniqueness result using WPO, SYM, HOM and PS. Addi on Σ^2 is clearly not available. It is interesting to note that our domain Σ^2 is naturally implied by the interesting discussion on Axiomatic Bargaining contained in Moulin [1983]. Moulin [1983], considers a domain which is a strict subset of Σ^2. However, all choice problems in Σ^2 can be obtained as the limit in the Hansdorff topology of a sequence of increasing choice problems considered by Moulin [1983].

3) Since $\bar{A}$ does not satisfy NIIA, the interesting axiomatic characterization on the subdomain of Σ^2 defined by

$\{S \in \Sigma^2 / u(S) = \{\bar{A}(S)\}\}$ using PO, SYM, TC and NIIA which is there in Exercise 3.9 of Moulin [1983] fails to generalize.

5 The Weighted Additive Choice Function

The conventional method of extending the additive choice function is to consider maximizers of the weighted sum of the coordinates. We however restrict ourselves to a particular kind of weighting system, so that the resulting solution satisfies STC.

Given $S \in \Sigma^2$, let $h(S) \equiv (h_1(S), h_2(S))$ where $h_i(S) = \max\{x_i / x \in S\}$, i = 1, 2. Clearly $h_i(S) > 0$, $i = 1, 2$, $\forall S \in \Sigma^2$.

Let $f_i(S) = \dfrac{1}{h_i(S)}$, $i = 1, 2$. Thus $h(f(S)S) = (1,1)$ $\forall S \in \Sigma^2$.

Let $\bar{B} : \Sigma^2 \to \mathbb{R}^2_+$ be a choice function defined as follows:

$$\bar{B}(S) = \left(h_1(S)\left[\frac{a_1(f(S),S) + b_1(f(S),S)}{2}\right], h_2(S)\left[\frac{a_2(f(S),S) + b_2(f(S),S)}{2}\right]\right)$$

where a_i, b_i, i=1, 2 are functions from Σ^2 to $\mathbb{R}_+$ defined earlier. Clearly $\bar{B}(S)$ corresponds to the expected value of the random vector which has a uniform distribution on the set

$$\left\{ x \in S / \frac{x_1}{h_1(S)} + \frac{x_2}{h_2(S)} \geq \frac{y_1}{h_1(S)} + \frac{y_2}{h_2(S)} \ \forall (y_1, y_2) = y \in S \right\}.$$

The particular case when this set reduces to $\{\bar{B}(S)\}$ is known as the choice function due to Cao [1981].

We now invoke the following axioms:

13) Restricted Convex Linearity (RC. LIN):

$\forall S,T \in \Sigma^2$ *with* $h(S) = h(T)$, $F(\alpha S + (1-\alpha)T) = \alpha F(S) + (1-\alpha)F(T)$ $\forall \alpha \in [0,1]$

14) Restricted Binary Additivity (RB. Addi):

$\forall S,T \in \Sigma^2$ with h(S) = h(T) and $F_1(S) + F_2(S) = F_1(T) + F_2(T)$, $F(V) = \frac{1}{2}[F(S) + F(T)]$ where

V = Comprehensive convex hull {S,T} provided

$u(S) = \{\bar{A}(S)\}$ and $u(T) = \{\bar{A}(T)\}$.

We thus have the following theorem:

Theorem 2:

The only choice function on Σ^2 to satisfy PO, STC, SYM, RC.LIN and RB.Addi is $\bar{B}$.

Proof:

If F satisfies the assumptions then $F = \bar{B}$ is easily established along the lines of the relevant part of the proof in Theorem 1, by setting h(V) = (1,1) (permissible by STC) and by noting that Cases 3 and 4 can thus not arise. The other way is easy to check. Q.E.D.

6 Conclusion

The main contribution of this paper is a definition of the additive choice function on an extended domain rather than on a severely restricted domain which existed earlier. However, earlier axiomatic characterizations carry over to our new framework. Thus we have now been able to define in a meaningful way the additive choice function and its scale translation covariant analogue and provide axiomatic characterizations for them as well.

Appendix

In this appendix and in view of Remark (2) (after Theorem 1), we prove an axiomatic characterization of the egalitarian choice function using the superadditivity axiom. We invoke the following two assumptions as well.

Strong Individual Rationality (SIR):

$F(S) \gg 0 \forall S \varepsilon \Sigma^2$

Continuity (CONT):

If $\{S^k\}$ be a sequence in Σ^2 converging to $S \varepsilon \Sigma^2$ in the Hansdorff topology, then

$\lim_{k \to \infty} F(S^k) = F(S)$.

We now prove the following theorem:

Theorem:

The only choice function on Σ^2 to satisfy SIR, WPO, SYM, NIIA, S.Addi and CONT is the egalitarian choice function E defined as follows:

$\forall S \varepsilon \Sigma^2$, $E(S) = (\bar{t}, \bar{t})$, where $\bar{t} = \max\{t/(t,t) \varepsilon S\}$.

To prove this theorem we use the following lemma:

Lemma:

Under the hypothesis of the theorem, $F(T) \geq E(T)$ $\forall T \varepsilon \Sigma^2$ of the form $T = \{x \varepsilon \mathbf{R}^2_+ / x \leq a\}$ for some $a \gg 0$.

Proof:

If $a = (a_1, a_2)$ with $a_1 = a_2$, then F(T) = E(T) by WPO and SYM.

Hence suppose W.l.o.g. $a_1 > a_2$.

Thus $E(T) = (a_2, a_2)$

Let $b(\varepsilon) = (1 - \varepsilon)\, a_2$ for $0 < \varepsilon < 1$.

$$T(\varepsilon) = \left\{x \in \mathbb{R}^2_+ / x \leq (b(\varepsilon),\ b(\varepsilon))\right\}$$

$$U(\varepsilon) = \left\{x - (b(\varepsilon),\ b(\varepsilon)) / x \geq (b(\varepsilon),\ b(\varepsilon))\ x \in T\right\}.$$

Then $T = T(\varepsilon) + U(\varepsilon)\ \forall\ 0 < \varepsilon < 1$.

$$\therefore F(T) \geq F(T(\varepsilon)) = (b(\varepsilon),\ b(\varepsilon))\ \forall\ 0 < \varepsilon < 1.$$

Taking limits as $\varepsilon \to 0$, we get $F(T) \geq E(T)$. Q.E.D.

Proof of Theorem:

That E satisfies the above properties is clear. Thus let us assume F satisfies the above properties and towards a contradiction assume that there exists $S \in \Sigma^2$ such that $F(S) \neq E(S)$. To begin with assume $E(S) \in P(S)$. The proof is completed by appealing to CONT.

Let T = Comprehensive convex hull $\{F(S)\}$

By NIIA, F(T) = F(S)

By Lemma above $F(T) \geq E(T)$

Clearly $F(T) \neq E(T)$ for then F(S) = E(S)

Without loss of generality assume $F_1(T) > E_1(T)$

Since $E(T) \in W(T)$, $F_2(T) = E_2(T)$

Let T' = Comprehensive convex hull $\{E(T)\}$

$$F(T') = E(T') = E(T)$$

Let $U = \left\{x - E(T) \in \mathbb{R}^2_+ / x \in S\right\}$

$U \in \Sigma^2$, since $E(S) \in P(S)$

$T' + U \subset S$ and $F(S) = F(T) \in U + T'$

By NIIA, $F(T' + U) = F(S) = F(T)$

But $F(T' + U) \geq F(T') + F(U)$ by S. Addi.

i.e. $F(T) \geq E(T) + F(U)$

By SIR, $F(U) > 0$

$$\therefore F(T) > E(T)$$

Contradicting $F_2(T) = E_2(T)$. Q.E.D.

In the above proof we invoke the Nash's Independence of Irrelevant Alternatives Assumption, which sets the egalitarian choice function apart both from the choice function of Perles and Maschler [1981] and the choice function that we define in this paper.

Further since, SIR + HOM + NIIA → WPO, the following collary is immediate:

Collary:

The only choice function on Σ^2 to satisfy SIR, HOM, NIIA, S. Addi, SYM and CONT is the egalitarian choice function.

Reference

N. Anbarci and J.P. Bigelow [1994]: "The area monotonic solution to the cooperative bargaining problem," *Mathematical Social Sciences* 28, 133-142.

X. Cao [1981]: "A class of solutions to two-person bargaining problems," Harvard University Mimeo.

Y. Chun [1988]: "The equal-loss principle for bargaining problems," *Economics Letters*, 26, 103-106.

Y. Chun and H. Peters [1988]: "The lexicographic egalitarian solution," *Cahiers du CERO*, 30, 149-156.

Y. Chun and H. Peters [1991]: "The lexicographic equal-loss solution," *Mathematical Social Sciences*, 22, 151-161.

E. Kalai [1977]: "Proportional solutions to bargaining situations: Interpersonal utility comparisons," *Econometrica*, 43, 513-518.

S. Lahiri [1996]: "Some remarks on properties for choice functions," mimeo, Indian Institute of Management, Ahmedabad.

H. Moulin [1983]: "Axioms of cooperative decision making," *Econometric Society Monograph*, Cambridge University Press.

R. Myerson [1981]: "Utilitarianism, egalitarianism, and the timing effect in social choice problems," *Econometrica*, 49, 883-897.

M.A. Perles and M. Maschler [1981]: "The super-additive solution for the Nash bargaining game," *International Journal of Game Theory*, 10, 163-193.

H. Peters [1986]: "Simultaneity of issues and additivity in bargaining," *Econometrica*, 54, 153-169.

PURCHASED
APPROVAL
GRATIS/EXCHANGE
PRICE
ACC NO.
VIKRAM SARABHAI LIBRARY
I. I. M. AHMEDABAD