W P : 273

Working Paper

LAGGED VARIABLE MODELS
AND THEIR ESTIMATION

by

G S Gupta
Deepak Chawle

INDIAN INSTITUTE OF MANAGEMENT
AHMEDABAD

LAGGED VARIABLE MODELS AND THEIR ESTIMATION

by

G S Gupta
Deepak Chawle

W P No. 273
April 1979

The main objective of the working paper series of the IIMA is to help faculty members to test out their research findings at the pre-publication stage.

INDIAN INSTITUTE OF MANAGEMENT
AHMEDABAD

LAGGED VARIABLE MODELS AND THEIR ESTIMATION

G.S. Gupta and Deepak Chawla*

There are three situations under which one of the crucial assumptions of the general linear model, that is

$$\text{CoV}(X_{it}, U_t) = 0, \text{ for } i = 1, 2, \ldots\ldots\ldots, k \quad t = 1, 2, \ldots\ldots\ldots, n \qquad \ldots\ldots (1)$$

(X_{it} is the i^{th} explanatory variable and U_t is the error term in observation t.)

is violated:

(a) there exists errors in the measurement of variables,

(b) at least one of the explanatory variables i a lagged dependent variable, and

(c) at least one of the explanatory variables is an endogenous (explained) variable.

The use of ordinary least squares (OLS) method to estimate models, in which the above assumption does not hold good, yields biased and inconsistent estimates.[1/] Although there is not yet any estimation procedure which could provide unbiased estimates under such a situation, there exists techniques to obtain consistent estimates. In this paper, the estimation methods for lagged variable models ~~are~~ discussed and they are illustrated through the estimation of consumption function for India.

---

*The authors are Professor and Doctoral student, respectively at the Indian Institute of Management, Ahmedabad.

1/ See Johnston, J. (1972): Econometric Methods, Chapters 9-10.

The lagged variable models are characterized by the presence of either lagged explanatory variables or/and lagged dependent variable. The example of the former is the distributed lag model, and of the latter are the partial adjustment model and the expectation model.[2/]

DISTRIBUTED LAG MODELS

The standard regression model specifies a causal relation between a dependent variable and one or more independent variables. Such a specification implies that a unit change in one of the independent variables caûses a change in the dependent variable during the same period and during that period alone. In some situations, however, this specification may seem restrictive. For example, advertisement outlay of a firm in a particular year affects its sales in the current as well as some future periods. Similarly, income of a given year might cause both the current and future consumptions to rise. In such causal relations, the total influence of a change in an independent variable is not felt in the same period that the cause occured but is distributed over time and thus the full reaction is evoked only after some passage of time, after some lag. Such models are called the distributed lag models. Mathematically,

---

2/ The Robertsonian lagged response model in which current consumption is hypothesized to depend on previous year's income does not violate the least squares assumption which renders the estimates biased and inconsistent.

a linear version of these models can be written as [3/]

$$Y_t = \alpha + \beta_o X_t + \beta_1 X_{t-1} + \cdots\cdots\cdots + \beta_s X_{t-s} + U_t \quad \ldots\ldots(2)$$

(Y is the dependent variable, X's are independent variables and s stands for the length of the lag).

Estimation of model (2) poses problems in terms of deciding the length of the lag, reduction in the degrees of freedom caused through increased number of explanatory variables and decreased number of observations, and multicollinearity. To avoid these problems, Koyck has suggested a device.[4] He assumes that the lag coefficients are a set of geometrically falling weights. Under this assumption, model (2) becomes

$$Y_t = \alpha + \beta_o X_t + \lambda^1 \beta_o X_{t-1} + \lambda^2 \beta_o X_{t-2} + \cdots\cdots\cdots + \lambda^s \beta_o X_{t-s} + U_t \quad \ldots\ldots (3)$$

$$(0 < \lambda < 1)$$

Lagging equation (3) by one time period and multiplying that by both sides, we have

---

3/ The model contained in equation (2) has the current and lagged values of only one variable as the explanatory variables. The other variables, current and/or lagged could easily be introduced. For simplicity alone, this has not been done here.

4/ Koyck, L.M. (1954): Distributed Lag and Investment Analysis, North Holland Publishing Co., Amsterdam.

$$\lambda Y_{t-1} = \lambda\alpha + \lambda\beta_o X_{t-1} + \lambda^2 \beta_o X_{t-2} + \lambda^3 \beta_o X_{t-3} + \dots + \lambda^{s+1} \beta_o X_{t-s+1} + \lambda U_{t-1} \quad \dots\dots (4)$$

Subtraction of (4) from (3) gives[5]

$$Y_t = \alpha(1-\lambda) + \beta_o X_t + \lambda Y_{t-1} + (U_t - \lambda U_{t-1})$$

or,

$$Y_t = \alpha^* + \beta_o X_t + \lambda Y_{t-1} + U_t^* \quad \dots\dots (5)$$

where

$$\alpha^* = \alpha(1-\lambda)$$

$$U_t^* = U_t - \lambda U_{t-1}$$

Thus, it is clear that the distributed lag model reduces to a lagged dependent variable model under Koyck's assumption. Equation (5) is in a manageable form for estimation.

PARTIAL ADJUSTMENT MODELS

Under a partial adjustment model, the behavioural equation determines the optimum (desired) value of the dependent variable and the adjustment equation gives the actual change in that variable over its previous value:[6]

$$Y_t^* = A + BX_t + U_t \quad \dots\dots (6)$$

$$Y_t - Y_{t-1} = \gamma (Y_t^* - Y_{t-1}) \quad \dots\dots (7)$$

5/ The term $\lambda^{s+1} \beta_o X_{t-s+1}$ has been ignored, for it is insignificant.

6/ As in equation (2) above, only one explanatory variable is considered in equation (6) below and in equation (9) latter for simplicity.

(Y_t^* stands for the optimum level of Y_t and γ for coefficient of adjustment such that $0 < \gamma < 1$)

The adjustment is partial because of ignorance, inertial and the cost of change. To illustrate, an investment that a firm might like to undertake may not materialize fully either because funds are not available, the physical asset that it aims to procure are not immediately available or/and that the firm is not sure as to whether the enhanced need for investment is of permanent nature.

Substitution of equation (6) in equation (7) yields

$$Y_t = \gamma A + \gamma B X_t + (1 - \gamma) Y_{t-1} + \gamma U_t$$

i.e., $$Y_t = A^* + B^* X_t + \gamma^* Y_{t-1} + U_t^* \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots \quad (8)$$

where $A^* = \gamma A; \quad B^* = \gamma B$

$\gamma^* = (1 - \gamma); \quad U_t^* = \gamma U_t$

Equation (8) is similar to equation (5) and it can be estimated in the present form through appropriate techniques.

EXPECTATION MODELS

Under the expectation models, the value of a dependent variable is determined in two stages: The first stage explains the variable in terms of the expected value of its cause variables and the second stage describes the formation of expectations. Mathematically, these could be put as:

$$Y_t = a + b\, X_t^e + U_t \qquad \text{..................(9)}$$

$$X_t^e = a_o\, X_{t-1} + a_1\, X_{t-2} + a_2\, X_{t-3} + \text{..................(10)}$$

(X_t^e stands for the expected value of X in period t)

The current value of Y depends on the expected rather than the true value of X, for the true value is not known at the time of decision-making. Besides, the expectations are formed on the basis of past values of the variable under question. To illustrate, the investment that a firm might like to undertake may depend on the profit it is expected to make, which, in turn, may be determined as a weighted average of the past profits.

Substitution of equation (10) in equation (9) yields

$$Y_t = a + a_0 b\, X_{t-1} + a_1 b\, X_{t-2} + a_2 b\, X_{t-3} + \cdots + U_t \qquad \text{.....(11)}$$

This equation is similar to equation (2) above, which describes a distributed lag model. If the weights of equation (10) are assumed to decline in geometrical progression, like equation (2) above which was reduced to equation (5), equation (11) can easily be reduced to the following form:

$$Y_t = a(1 - \delta) + a_o b\, X_{t-1} + \delta Y_{t-1} + (U_t - \delta U_{t-1})$$

$$(o < \delta < 1)$$

or $$Y_t = a^* + b^*\, X_{t-1} + \delta Y_{t-1} + U_t^* \qquad \text{.............(12)}$$

where $a^* = a(1 - \delta)$

$b^* = a_0 b$

$u_t^* = (u_t - \delta u_{t-1})$

Equation (12) is similar to equation (5), the only difference is that instead of X_t, there is X_{t-1}.

ESTIMATION

To illustrate the estimation procedure of the lagged variable models, the consumption function for India has been estimated using the annual time series data of 1950-51 through 1975-76 (Vide Table 1). The estimation is carried out through three methods, a discussion of these follows:[7/]

Estimation through Ordinary Least Squares Method:

The OLS estimates of the simple consumption function and of that with the lagged variable are the following:

$$C_t = \underset{(2.01)}{1647.06} + \underset{(14.39)}{0.794}\, Y_t \qquad \text{.......... (13)}$$

$$R^2 = 0.896, \quad DW = 1.75$$

7/ The direct estimation of models such as that contained in equation (2) poses problems of choosing the length of the lag, multicollinearity and loss of degrees of freedom. To overcome these, suggestions have also been made to reduce the current and all lagged values of a variable to one variable by assigning a priori weights to these. However, this is not an appropriate method of estimation, for if weights can be known a priori, coefficients values can also be decided without estimating them.

Table 1: Data for Consumption Model

(Rs. in crores)

Year	Private Disposable Income at 1960-61 Prices (Y_t)	Private Consumption Expenditure at 1960-61 Prices (C_t)	Estimated Private Real Consumption lagged one period (C_{t-1})
(1)	(2)	(3)	(4)
1950-51	8,926	8,518	
1951-52	9,164	8,837	
1952-53	9,558	9,243	
1953-54	10,172	9,672	
1954-55	10,392	9,647	
1955-56	10,638	9,680	9,589
1956-57	11,149	10,221	9,867
1957-58	11,026	10,360	10,192
1958-59	11,797	11,030	10,626
1959-60	12,027	11,154	10,765
1960-61	12,727	12,210	11,135
1961-62	13,151	12,205	11,465
1962-63	13,324	12,346	11,757
1963-64	19,935	12,767	12,278
1964-65	15,165	13,565	12,558
1965-66	14,475	13,084	14,828
1966-67	14,774	13,629	14,395
1967-68	16,190	15,424	14,585
1968-69	16,701	14,927	16,366
1969-70	17,555	15,631	14,979
1970-71	18,377	16,756	15,084
1971-72	18,593	17,454	15,696
1972-73	18,454	17,077	16,628
1973-74	19,472	17,833	17,089
1974-75	19,423	17,717	17,498
1975-76	21,339	19,043	18,110

Sources: 1. India: Central Statistical Organization: National Accounts Statistics, January, 1975.

2. India: Central Statistical Organization: Statistical Abstract of India (Various Issues).

Note: The data are the revised series as brought out recently by C.S.O. Since, the revised series on C & Y for the period 1950-51 to 1959-60 was not available, the same was interpolated. The average relationship between the revised and traditional data was computed and the same was applied to the corresponding traditional series to generate revised series.

$$C_t = \underset{(0.44)}{209.83} + \underset{(2.02)}{0.160}\, Y_t + \underset{(8.50)}{0.834}\, C_{t-1} \qquad \ldots\ldots\ldots(14)$$

$$R^2 = 0.974, \qquad DW = 2.66$$

where

C_t = Private consumption expenditure at 1960-61 prices in period t.

Y_t = Private disposable income at 1960-61 prices in period t.

R^2 = Coefficient of Determination

DW = Value of the Durbin-Watson statistic
Numbers in parentheses, here and hereafter, are the corresponding t-values.

Under equation (13), the marginal propensity to consume (MPC) is 0.794, while in function (14) the short-run MPC is 0.160, and the long-run MPC is 0.964 $\left(\frac{0.160}{1 - 0.834}\right)$. If equation (14) is based on the distributed lag model, or the expectation formation model, its original form under Koyck's assumption would be

$$C_t = 1264.04 + 0.160\, Y_t + 0.133\, Y_{t-1} + 0.111\, Y_{t-2} + 0.926\, Y_{t-3} + \ldots. \qquad \ldots\ldots\ldots\ldots (15)$$

It should be noted that equation (13) disregards the influence of past on current consumption and the estimates of equation (14) and hence of equation (15) are inconsistent, for C_{t-1} is correlated with U_t, the disturbance term of the function.

Estimation Through Liviatan Method:

Under the Liviatan method, estimates are obtained in two stages. In the first stage, the current value of the stochastic variable (C_t) is regressed on lagged values of the non-stochastic explanatory variable. The result of this step for the consumption function are as follows:[8/]

$$\hat{C}_t = \underset{(1.29)}{961.45} + \underset{(2.88)}{0.293}\, Y_{t-1} + \underset{(1.11)}{0.119}\, Y_{t-2} + \underset{(1.25)}{0.135}\, Y_{t-3} + \underset{(3.58)}{0.366}\, Y_{t-4} \qquad \text{............ (16)}$$

$$R^2 = 0.950, \quad DW = 0.86.$$

In the second stage of the Liviatan's method, the function is estimated by the OLS method but instead of using the true values of the stochastic explanatory variable, its estimated values, as obtained in the first stage, are used. The values of $\hat{C}_{t-1}$ as derived from equation (16) are given in Table 1, Column 4. The results of this step are the following:

$$C_t = \underset{(0.30)}{258.44} + \underset{(2.70)}{0.286}\, Y_t + \underset{(5.30)}{0.684}\, \hat{C}_{t-1} \qquad \text{..........(17)}$$

$$R^2 = 0.936, \quad DW = 1.99$$

The R^2 value of this function is not quite meaningful, for $\hat{C}_{t-1}$ instead of C_{t-1} is used in theestimation procedure. The R^2 obtained by using these estimates of the parameters and replacing back $\hat{C}_{t-1}$ by C_t came to 0.931.

Under function (18), the short-run MPC is 0.286 and the long-run MPC is 0.905 $\left(\frac{0.286}{1 - 0.684} \right)$. The original form of equation (18) under

---

8/ Lagged values upto four periods alone were used, for inclusion of more distant values did not improve the R^2 value.

the Koyck's assumption would be

$$C_t = 817.85 + 0.286\, Y_t + 0.196\, Y_{t-1} + 0.134\, Y_{t-2} + 0.092\, Y_{t-3} + \ldots \qquad \ldots (18)$$

The parameter estimates of equations (17) and (18) are consistent, for $\hat{C}_{t-1}$ is uncorrelated with U_t.

Estimation Through Almon's Method:

Almon's method provides estimates of the distributed lag models and the expectation models in their original forms as in equations (2) and (11), respectively. The method assumes that the β coefficients are related through the following polynomial:

$$\beta_z = a_0 + a_1 Z + a_2 Z^2 + \ldots + a_r Z^r \qquad \ldots (19)$$

Several values of s and r are tried and their appropriate combination is chosen in the estimation process. For r = 1, equation (19) becomes

$$\begin{aligned} \beta_0 &= a_0 \\ \beta_1 &= a_0 + a_1 \\ \beta_2 &= a_0 + 2a_1 \\ \beta_3 &= a_0 + 3a_1 \\ &\vdots \\ \beta_z &= a_0 + Za_1 \end{aligned} \qquad \ldots (20)$$

Substitution of these values in equation (2), writing C for Y and Y for X, and assuming s = 3 yields

$$C_t = \alpha + a_0 Y_t + (a_0 + a_1) Y_{t-1} + (a_0 + 2a_1) Y_{t-2} + (a_0 + 3a_1) Y_{t-3} + U_t$$

or, $$C_t = \alpha + a_0 (Y_t + Y_{t-1} + Y_{t-2} + Y_{t-3}) + a_1 (Y_{t-1} + 2Y_{t-2} + 3Y_{t-3}) + U_t$$

i.e., $$C_t = \alpha + a_0 Y^0 + a_1 Y^1 + U_t \qquad \text{.......... (21)}$$

where, $$Y^0 = Y_t + Y_{t-1} + Y_{t-2} + Y_{t-3}$$

$$Y^1 = Y_{t-1} + 2Y_{t-2} + 3Y_{t-3}$$

Equation (21) is amenable to estimation through the OLS method. The OLS estimates of equation (21) obtained on the basis of the data of Table 1 are the following

$$\underset{}{C_t} = \underset{(1.09)}{792.49} + \underset{(3.80)}{0.288\, Y^0} - \underset{(0.87)}{0.044\, Y^1} \qquad \text{......(22)}$$

$$R^2 = 0.943, \quad DW = 1.03.$$

Converting this equation in the original form of the distributed lag consumption function, using the parameters' definitions as in equations (20) and (21), yields

$$C_t = 792.49 + 0.288\, Y_t + 0.244\, Y_{t-1} + 0.200\, Y_{t-2} + 0.156\, Y_{t-3} \qquad \text{.............(23)}$$

Equation (23) provides the Almon's estimates of the consumption function. The value of R^2 for these estimates of parameters comes to 0.94. Under this, the short-run (the first period) MPC is 0.288, and the long-run MPC is 0.888 (0.288 + 0.244 + 0.200 + 0.156). The Almon's estimates are consistent.

The readers might wonder as to how to choose the appropriate values of s and r for using the Almon's method. For this, the researcher needs to choose alternative combinations of their values and estimate as many functions as the alternative combinations of the resulting equations like (22). Of the various combinations, the one which yields the best (on theoretical and statistical grounds) form for the intermediate equation like (22) and the final equation like (23), is to be selected.

CONCLUSION

Equations (15), (18) and (23) provide alternative estimates of the consumption function for India. Obviously, the results are not identical and there is a problem of choice. Equation (15) is unacceptable on theoretical grounds, and whether equation (18) or equation (23) should be selected depends upon the researcher's belief about their assumptions. If he believes that the effect of distant values of incomes on consumption should be less than that of the immediate past levels of income and that this effect must decline in geometric progression, he would prefer equation (18) to equation (23). In contrast, if he does not have any belief about the relative effects of past levels of incomes on consumption but he thinks that these effects follow a polynomial of type given in equation (19), then he would select equation (23) as his estimated consumption function.

It is pertinent to note here that the Almon's method is not available for estimating a partial adjustment model. Further, this method need not give declining values for β coefficients over the lagged values of the explanatory variable. It g. es either declining or increasing coefficients and the change is in arithmetical progression. In equation (22), if the coefficient of Y^1 were positive, the successive β coefficients would have been larger than the previous ones in equation (23). In the model estimated, the Almon's coefficients (vide equation (23)) take declining values, though in arithmetical rather than in geometrical progression, and so the estimates in equations (18) do not differ with those in equation (23) substantially. In particular, the short-run (0.286 and 0.288) and long-run (0.905 and 0.888) MPC in both these functions are almost exactly the same and the estimates seem to be reasonable in the Indian context. This renders the choice between the two estimates redundant. Thus, we conclude that the short-run MPC is 0.29 and long-run MPC is 0.90 in India.

– – –

PRG