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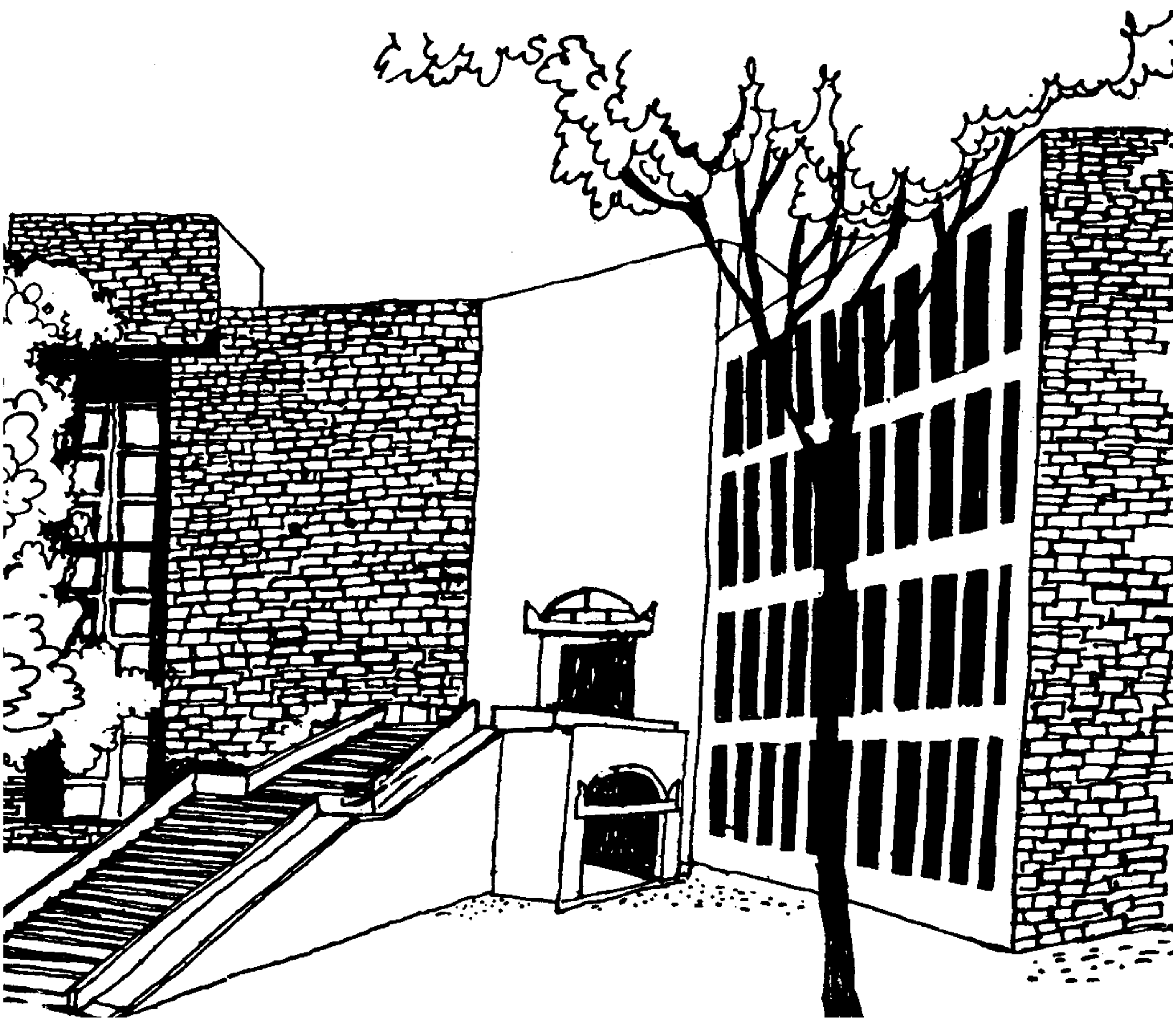

Monotonicity of Compromise Solutions With Respect to the Claims Point

(Revised)

Somdeb Lahiri
Indian Institute of Management
Ahmedabad

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Abstract

In this paper we establish that a family of well-known normed compromise solutions for two-person claims problems respond appropriately to changes in the claims point.

1. **Introduction** :- In this paper we shall be primarily interested in establishing that some well-known compromise solutions for claims problems respond appropriately to changes in the claims point. We shall confine our discussion primarily to the two-person case.

Following Young (1987), Yu (1973), Friemer and Yu (1976) and more recently Thomson (forthcoming) we define a two-person claims problem in the following fashion :

Let $S \subseteq \mathbb{R}^2$ be compact and convex; let $PO(S) \equiv \{x \in S / \forall y \in S\ [y \geqq x \Rightarrow y=x]\}$ denote the set of Pareto-optimal points of S; and let $c \in \mathbb{R}^2$ be such that there exists $x \in S$ for which $c \gg x$. Then the ordered pair (S,c) will be called a claims problem. Let Σ denote the class of claims problems as defined above. Richter (1982) considers a similar class of claims problems.

A compromise solution on Σ is a function $F: \Sigma \rightarrow \mathbb{R}^2$ such that $\forall (S,c) \in \Sigma$, $F(S,c) \in S$.

Yu (1973), Friemer and Yu (1976) and Richter (1982) consider the following class of compromise solutions:

Let $p \in [1, \infty]$. Following Thomson (forthcoming) we define $Y^p: \Sigma \rightarrow \mathbb{R}^2$ as

$$Y^p(S,c) = \text{argmin } \{(c_1-x_1)^p + (c_2-x_2)^p / (x_1,x_2) \in S,\ x_i \leqq c_i,\ i=1,2\}$$

for $1 \leqq p < \infty$.

$$Y^\infty(S,c) = \text{argmin } \{\max[c_1-x_1, c_2-x_2] / (x_1,x_2) \in S,\ x_i \leqq c_i,\ i=1,2\}.$$

These are conventionally referred to as the Yu-p solutions and the equal loss solution respectively.

Thomson (1987) studies, the appropriate responsiveness of the Nash (1950), Kalai-Smorodinsky (1975) and the Egalitarian (Kalai (1977)) solutions to certain unilateral changes in the disagreement point, for a fixed feasible set. Our purpose in this paper is to establish similar results for the family $\{Y^p: p \in [1,\infty]\}$ under unilateral changes in the claims point.

2. The Results :- We start by formulating our condition of monotonicity with respect to the claims point:

<u>c-monotonicity</u> (c-mon) : For all (S,c), (S',c'), for all i, if $S'=S$, $c_i' > c_i$ and $c_j'=c_j$ for all $j \neq i$, then $F_i(S',c') \geqq F_i(S,c)$.

It may be observed that in the two-person case if $F(S,c) \in PO(S) \forall (S,c) \in \Sigma$, then (c-mon) is equivalent to the following property:

<u>Strong c-monotonicity</u> (st. c-mon) : For all (S,c), (S',c'), for all i, if $S'=S$, $c_i' > c_i$ abd $c_j'=c_j$ for all $j \neq i$, then $F_j(S',c') \leqq F_j(S,c)$.

Before we proceed to the main results of this paper, let us gather together some important conclusions available with regard to the family $\{Y^p: p \in [1,\infty]\}$.

<u>Theorem 1</u> :- (a) For $1<p<\infty$, $Y^p:\Sigma \to \mathbb{R}^2$ is well-defined

(b) For $p=1,\infty$, $Y^p:\bar{\bar{\Sigma}} \to \mathbb{R}^2$ is well-defined where $\bar{\bar{\Sigma}} \equiv \{(S,c) \in \Sigma / S$ is strictly convex$\}$.

<u>Pf</u> :- Property 4.5 of Yu (1985).

<u>Theorem 2</u> :- (a) For $1<p<\infty$, $\forall (S,c) \in \Sigma$, $Y^p(S,c) \in PO(S)$

(b) For $p=1,\infty$, $Y^p(S,c) \in PO(S) \; \forall \; (S,c) \in \bar{\bar{\Sigma}}$.

The following property is significant for subsequent analysis.

<u>Continuity</u> (cont) : If $S^k \to S$ (in the Hausdorff topology) and $c^k \to c$ (in the Euclidean topology) then $F(S^k,c^k) \to F(S,c)$ where $\{(S^k,c^k)\}$ is an arbitrary sequence of claims problems.

Theorem 3 :- (a) For $1<p<\infty$, $Y^p:\Sigma \to \mathbb{R}^2$ satisfies continuity.

(b) For $p=1,\infty$, $Y^p:\bar{\bar{\Sigma}} \to \mathbb{R}^2$ satisfies continuity.

Proof :- The proof follows immediately from the definition of $\{Y^p: p\in[p,\infty]\}$ and the maximum theorem (see Berge (1962) or Lahiri (1990)).

In the sequel we well require the following subdomain of $\bar{\bar{\Sigma}}$ (see Thomson (1981)).

Given p in $\mathbb{R}^2$, $\|p\|$ denotes $|p_1| + |p_2|$.

$\Delta \equiv \{p\in\mathbb{R}^2_+ / \|p\|=1\}$.

Given $S\subseteq\mathbb{R}^2$, which is compact and convex and x in S,

$W(S,x) = \{p\in\mathbb{R}^2_+ / \forall y\in S,\ p.y, \leqq p.x\}$.

Note that for all x in the interior of S, $W(S,x) = \phi$ and for all $x\in PO(S)$, $W(S,x) \neq \phi$ by the separation theorem for convex sets (see Rockafellar (1972)). Define,

$\bar{\bar{\Sigma}}_{dif} \equiv \{(S,c)\in\bar{\bar{\Sigma}} / \forall x\in PO(S),\ W(S,x)$ contains atmost one point$\}$

Lemma 1 :- For $1<p<\infty$, $Y^p: \bar{\bar{\Sigma}}_{dif} \to \mathbb{R}^2$ satisfies c-mon.

Proof :- Let $(S,c)\in\bar{\bar{\Sigma}}_{dif}$. Then $\exists\ \underline{x},\bar{x}\in\mathbb{R}$ and $\phi:[\underline{x},\bar{x}] \to \mathbb{R}$ such that $PO(S) = \{(x_1,x_2)\in\mathbb{R}^2 / x_2=\phi(x_1),\ x_1\in[\underline{x},\bar{x}]\}$.

Further ϕ is differentiable, $\phi'<0\ \forall\ x_1\in[\underline{x},\bar{x}]$ and $\phi':[\underline{x},\bar{x}]\to\mathbb{R}$ is a non-increasing function.

Now, $(S,c)\in\bar{\bar{\Sigma}}_{dif}$ and $Y^p(S,c)=(x_1^*,x_2^*)\in S$ implies by Theorem 2, $x_2^* = \varphi(x_1^*)$ and by the definition of Y^p,

$(x_1^*,x_2^*) = \text{argmax}\ \{[c_1-x_1]^p+[c_2-x_2]^p / x_1,x_2)\in S,\ x_i\leqq c_i, i=1,2\}$.

Appealing to Theorems 1 and 2 we can assert that

x_1^* solves

$[c_1-x_1]^p+[c_2-\varphi(x_1)]^p \to \min$

s.t. $x_1\in[\underline{x},\bar{x}]$

The first order necessary and sufficient condition for x_1^* to solve the above problem is that

$$\left[\frac{c_1 - x_1^*}{c_2 - \varphi(x_1^*)}\right]^{p-1} = -\varphi'(x_1^*)$$

Now suppose, $(S,c') \in \bar{\Sigma}_{dif}$ where $c'_1 > c_1$ and $c'_2 = c_2$. For $(\tilde{x}_1, \varphi(\tilde{x}_1))$ to be equal to $Y^p(S,c')$ it is necessary and sufficient that

$$\left[\frac{c'_1 - \tilde{x}_1}{c_2 - \varphi(\tilde{x}_1)}\right]^{p-1} = -\varphi'(\tilde{x}_1).$$

Suppose towards a contradiction $\tilde{x}_1 < x_1^*$. Since $\varphi' < 0$, we have $\varphi(\tilde{x}_1) > \varphi(x_1^*)$.

$$\therefore c_2 - \varphi(\tilde{x}_1) < c_2 - \varphi(x_1^*)$$

Also $c'_1 - \tilde{x}_1 > c_1 - x_1^*$.

$$\therefore \left[\frac{c'_1 - \tilde{x}_1}{c_2 - \varphi(\tilde{x}_1)}\right]^{p-1} > \left[\frac{c_1 - x_1^*}{c_2 - \varphi(x_1^*)}\right]^{p-1}$$

i.e. $\varphi'(\tilde{x}_1) < \varphi'(x_1^*)$,

contradicting φ' is a non-increasing function and proving the lemma.

Q.E.D.

As a consequence of Lemma 1 and Theorem 3, we have the following theorem:

<u>Theorem 4</u> :- For $1 < p < \infty$, $Y^p : \Sigma \to \mathbb{R}^2$ satisfies c-mon.

<u>Pf</u> :- Let (S,c) and $(S,c') \in \Sigma$ and let $c_1 < c'_1$, $c_2 = c'_2$. There exists sequences $\{(S^k,c)\}_{k=1}^{\infty}$ and $\{(S^k,c')\}_{k=1}^{\infty}$ of claims problems in Σ_{dif} such that $S^k \to S$ is the Hausdorff topology.

By Lemma 1, $Y_1^p(S^k,c') \geq Y_1^p(S^k,c) \; \forall k = 1,2,\ldots$

By Theorem 3,

$$\lim_{k\to\infty} Y^p(S^k, c') = Y^p(S, c')$$

$$\lim_{k\to\infty} Y^p(S^k, c) = Y^p(S, c).$$

Combining the two results we get,

$$Y_1^p(S, c') \geqq Y_1^p(S, c)$$

which proves the theorem.

Q.E.D.

The two remaining cases for which c-mon requires to be proved are p=1 and $p=\infty$. For p=1, the c-monotonicity is obvious, since the solution is independent of the claims point.

This may be summarized in the following theorem:

<u>Theorem</u> <u>5</u> :- $Y^1 : \overline{\overline{\Sigma}} \to \mathbb{R}^2$ satisfies c-mon.

<u>Proof</u> :- Direct.

The only case that remains to be tackled is the case when $p=\infty$. In this case the proof is analogous to the proof of Theorem 3 in Thomson (1987). Let $m(S)=(m_1(S), m_2(S))$, $m_i(S)=\min\{x_i / x\in S\}$ $\forall (S,c)\in\overline{\overline{\Sigma}}$.

<u>Theorem</u> <u>6</u> :- $Y^\infty : \overline{\overline{\Sigma}}_- \to \mathbb{R}^2$ satisfies c-mon, where $\overline{\overline{\Sigma}}_- = \{(S,c)\in\overline{\overline{\Sigma}} / m(S) \leqq y \leqq x,\ x\in S \Rightarrow y\in S\}$.

<u>Proof</u> :- Suppose by way of contradiction, that for some (S,c) and $(S,c')\in\overline{\overline{\Sigma}}_-$ with $c_1' > c_1$, and $c_2' = c_2$, we have $Y_1^\infty(S,c') < Y_1^\infty(S,c)$. From the definition of Y^∞ it follows that

$$c_1' - Y_1^\infty(S,c') = c_2 - Y_2^\infty(S,c')$$

$$c_1 - Y_1^\infty(S,c) = c_2 - Y_2^\infty(S,c).$$

$$\begin{aligned} \therefore\ Y_2^\infty(S,c) - Y_2^\infty(S,c') &= [c_2 - Y_2^\infty(S,c')] - [c_2 - Y_2^\infty(S,c)] \\ &= [c_1' - Y_1^\infty(S,c')] - [c_1 - Y_1^\infty(S,c)] \\ &= [c_1' - c_1] + [Y_1^\infty(S,c) - Y_1^\infty(S,c')] > 0 \end{aligned}$$

$\therefore Y_2^{\infty}(S,c) > Y_2^{\infty}(S,c')$ and thus $Y^{\infty}(S,c) >> Y^{\infty}(S,c')$, contradicting the Pareto optimality of Y^{∞}.

Q.E.D.

Remark : We could have alternatively defined,

$$Y^{\infty}(S,c) = c-[\alpha(S,c),\alpha(S,c)] \; \forall \; (S,c) \in \Sigma_{-}$$

where,

$$\Sigma_{-} = \{(S,c) \in \Sigma / m(S) \leqq y \leqq x, \; x \in S \Rightarrow y \in S\}$$

and $\alpha(S,c) = \min \{\alpha \geqq 0 / c-(\alpha,\alpha) \in S\}$.

It is easy to check that Y^{∞} as enunciated here is well defined and for all $(S,c) \in \Sigma_{-}$, $Y^{\infty}(S,c)$ is a weakly Pareto optimal point of S (i.e. there does not exist $x \in S$ such that $x >> Y^{\infty}(S,c)$). Further the analog of Theorem 6 would read : $Y^{\infty}: \Sigma_{-} \to \mathbb{R}^2$ satisfies c-mon. The proof of this assertion is identical to that of Theorem 6 in the paper, adapted to the appropriate domain.

Conclusion : In this paper, we have succeeded in showing that a class of well-known solutions for two-person claims problems satisfy c-monotonicity. The fact that these solutions satisfy what may be considered an intuitively desirable property, reinforces their importance. If we bear in mind, the use of these solutions in the study of normative taxation policies, the implication of c-monotonicity becomes clear.

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