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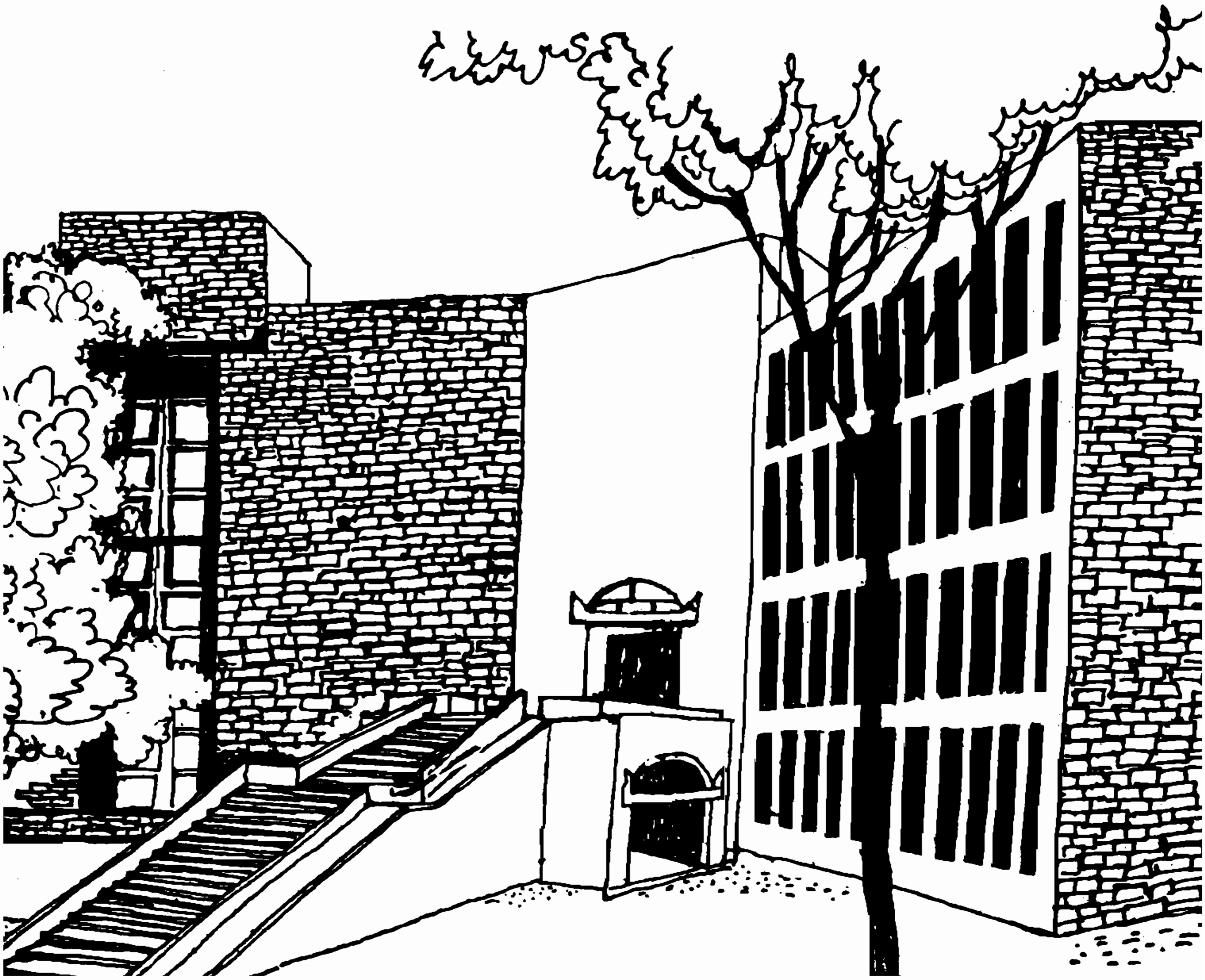

CUTTING PLANE BASED METHODS FOR INTEGER PROGRAMS

By

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Cutting Plane Based Methods for Integer Programs

1. Introduction

A wide variety of integer programs are NP-complete, i.e., they belong to the class of hard combinatorial optimization problems, and there is no known algorithm to date to solve them in polynomial time. Examples are the multi item lotsizing problems, with or without start-up costs and/or backlogging, plant location problems and the fixed charge network design problem. In all cases both the uncapacitated and capacitated versions of the problem are NP-complete, although capacity constraints often make the problem harder to solve.

These problems are standard in the operations research/management science literature not only because of their intrinsic theoretical content, but also because of their close similarity to real life problems. For instance, the fixed charge network problem frequently arises in a variety of contexts, including transportation, communication and distribution systems. Francis and Goldstein (1974) provide a detailed bibliography for the plant location problem, Minoux(1989) gives a detailed survey of network synthesis problems, and Magnanti and Vachani(1990) give a brief survey for the lot-sizing problems.

Various methods have been used to solve these problems. The running times of exact combinatorial algorithms varies exponentially with the size of input data, and hence they are not useful for problems of large size. Lagrangean relaxation methods have also not performed well on these class of problems, except for the plant location problem (see for instance Karmarkar and Schrage (1985)). Some dual ascent based methods have performed reasonably well in some cases (see for instance Wong (1984)), and also a dual decomposition based method by Magnanti, Wong and Mireault (1986). However, the success of cutting plane based methods in a variety of problems including the travelling salesman problem (Groetschel and Padberg (1979), Crowder and Padberg (1980), Padberg and Hong (1980), and Crowder, Johnson and Padberg (1983)), the lot-sizing problem (Barany, Van Roy and Wolsey (1984a and b), Leung, Magnanti and Vachani (1989), Wolsey (1989), Magnanti and Vachani(1990), and Sastry (1990)) has motivated research based on these methods for integer programs. The network design problem has not been studied very extensively using the cutting plane method and a lot of research needs to be done in this area. We have various versions of this problem including the single and multi commodity versions, with or without capacity constraints, with simultaneous or non-simultaneous flow, and with different types of cost structures. The Steiner tree problem is also useful in computer communication networks when trying to decide locations for concentrators used in routing messages. Although a specific application may have additional side constraints (for instance, capacities for communication lines in a telephone or computer network might be available in certain discrete sizes only), an understanding of the basic model will help in developing solution methods for specific instances.

2. A general framework.

We study only the uncapacitated versions of the problem. The capacitated version needs to be studied separately, and is outside the scope of this paper.

We describe the fixed charge, single commodity network design problem, and then show how the lot-sizing and plant location problems are special cases of the network design problem. Thus, one motivation for our study is to generalize results already obtained for the special cases to the more general network design problem.

The single commodity fixed charge problem can be formulated as follows. Let G = (V,A) be a directed graph with node set V and arc set $A \subseteq V \times V$. At each node there is a demand d_i if $d_i > 0$, or a supply d_i if $d_i < 0$. There is a fixed cost f_{ij} of installing arc (i,j), and a variable cost c_{ij} of sending one unit of flow from node i to j. Let x_{ij} denote the flow from node i to j, and let $y_{ij}=1$ if $x_{ij}>0$.

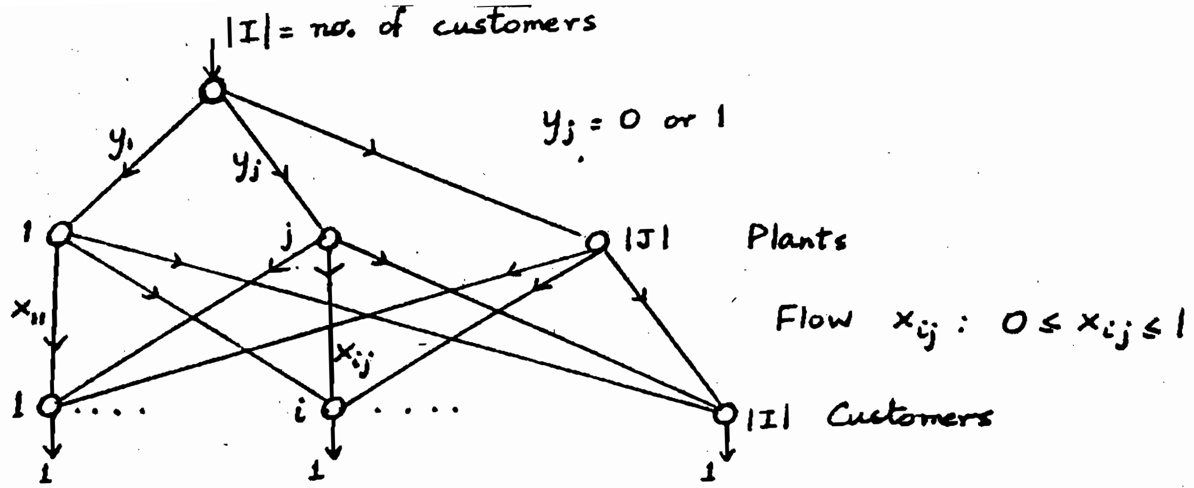
$$\min \{ \Sigma_{(i,j)\in A} c_{ij} x_{ij} + \Sigma_{(i,j)\in A} f_{ij} y_{ij} \} : (x, y) \in S$$

where S is the polyhedron

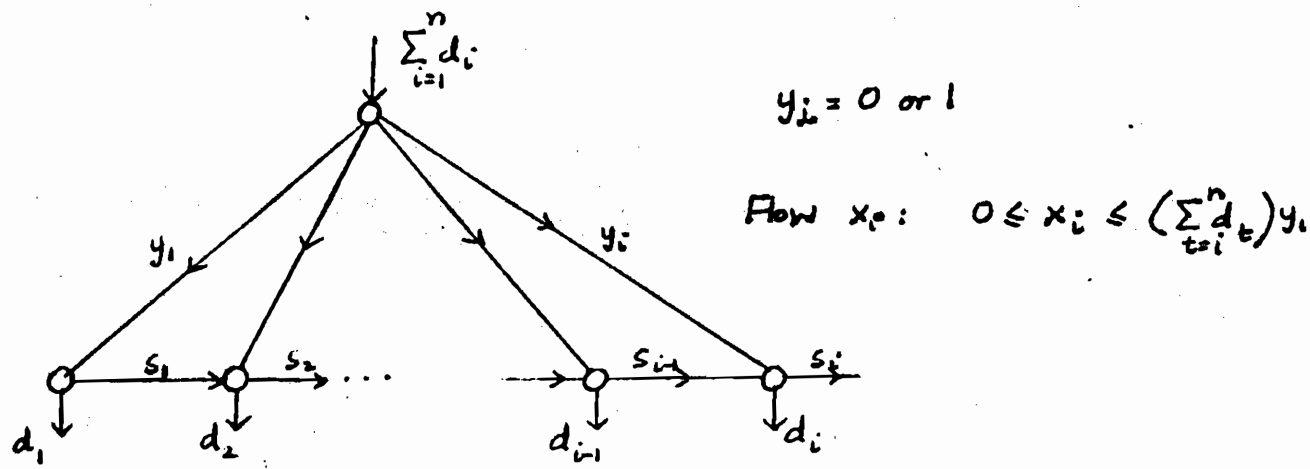
$$\Sigma_k x_{ik} - \Sigma_k x_{ki} = d_i \qquad \text{for all } i \in V$$
$$x_{ij} \le M y_{ij} \qquad \text{for all } (i,j) \in A$$
$$y_{ij} \le 1 \qquad \text{for all } (i,j) \in A$$
$$y \text{ integer and } x, y \ge 0.$$

M is a positive number with magnitude at least $\Sigma_{i\in V} d_i^+$ where $a^+ = \max(a, 0)$. Let LS denote the polyhedron obtained by relaxing the integer constraint on y.

Van Roy and Wolsey (1985) study this model and describe a class of valid inequalities for this problem, and show how to solve the separation problem for some special cases. Balakrishnan (1987) studied the multi-commodity problem with fixed origin-destination pairs for each commodity, and describes valid cuts for the problem. Minoux (1989) gives a comprehensive survey of network synthesis problems. The lot-sizing and plant location problems can be cast as follows:

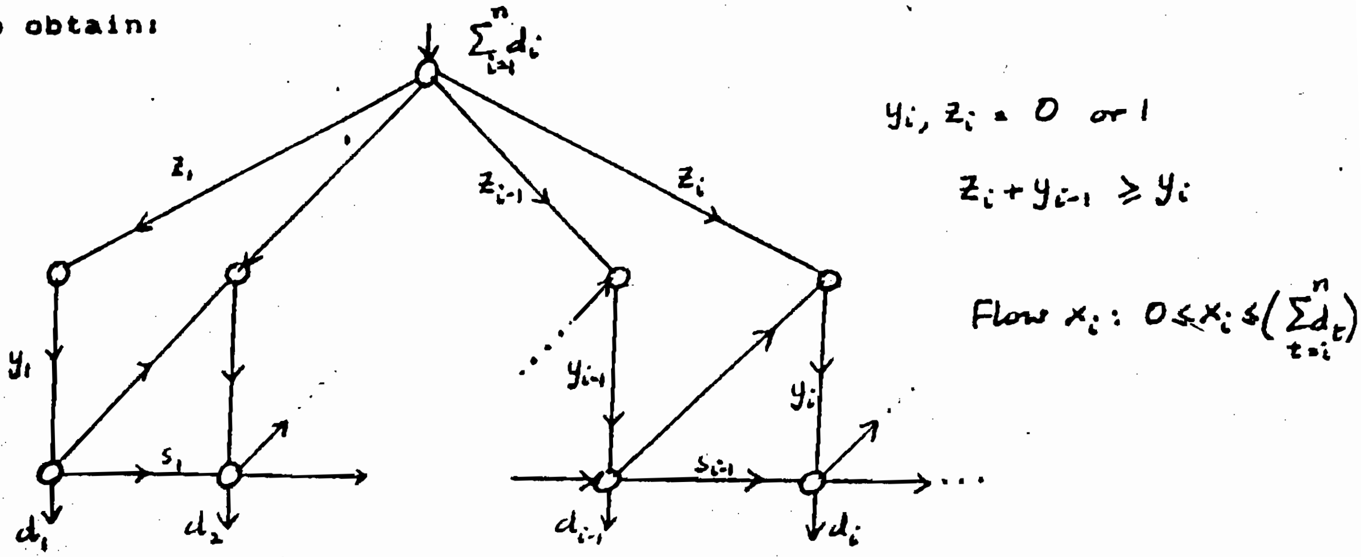


PLANT LOCATION



LOT- SIZING

If we extend the lot-sizing problem to include an additional start-up cost if we set up the machine in period i but not in period i-1, we obtain:



LOT-SIZING WITH START-UPS

3. Valid Inequalities

We describe a class of valid inequalities for the network design problem. For $N \subseteq V$, let $E=\{(i,j)\in A: i,j\in N\}$, $(N, V\backslash N) =\{(i,j)\in A: i\in N, j\in V\backslash N\}$, $C\subseteq(V\backslash N, N)$, $Q\subseteq E$. Then

$$\Sigma_{(i,j)\in C}x_{ij} \le \Sigma_{(i,j)\in C}a_{ij}y_{ij} + \Sigma_{(i,j)\in E\backslash Q}x_{ij} + \Sigma_{(i,j)\in (N,V\backslash N)}x_{ij} + K$$

is a valid inequality (VI) where a_{ij} and K are determined as follows:

Let

$J = \{j: (i,j)\in C\}$,

$V(Q) = J\cup\{k\in N$: there exists a path from $j\in J$ to k using only arcs of $Q\}$,

$D = \Sigma_{k\in V(Q)}d_k^+$

$V(j) = j\cup\{k\in N$: there exists a path from j to k using only arcs of Q, and there does not exist any path from $k'\neq k$, $k'\in J$ to node k using arcs from $Q\}$.

Theorem 1. The inequalities (VI) are valid for $a_{ij} = \Sigma_{k\in V(j)}d_k^+$ and $K = D - \Sigma_{j\in J}\Sigma_{k\in V(j)}d_k^+$.

Proof.

If $y_{ij} = 0$ for all $(i,j)\in C$, then $x_{ij} = 0$ for all $(i,j)\in C$ and the inequality is satisfied. Otherwise, let $J^* = \{j: y_{ij} = 1$ for some $(i,j)\in C\} \neq \Phi$. The flow from $V\backslash N$ to N along arcs in C either satisfies demand for nodes in V(Q), or flows along arcs in $E\backslash Q$, or comes back to nodes in $V\backslash N$ along arcs in $(N, V\backslash N)$. Consider the portion of the flow x'_{ij} for $(i,j)\in C$ that satisfies demand for nodes in V(Q).

Let

$V(Q^*) = J^*\cup\{k\in N$: there exists a directed path from some node $j\in J^*$ to k using only arcs of $Q\}$.

$D(J^*) = \Sigma_{k\in V(Q*)}d_k^+$.

Clearly,

$$\Sigma_{(i,j)\in C}x'_{ij} = \Sigma_{j\in J^*}\Sigma_{\{i:(i,j)\in C\}}x'_{ij} \leq D(J^*).$$

But since $y_{ij} = 1$ for $j\in J^*$, and $y_{ij} = 0$ for $j\in J\backslash J^*$,

$$\Sigma_{(i,j)\in C}a_{ij}y_{ij} = \Sigma_{j\in J^*}\Sigma_{\{i:(i,j)\in C\}}a_{ij} \geq \Sigma_{j\in J^*}\Sigma_{k\in V(j)}d_k^+.$$

Moreover, $K + \Sigma_{j\in J^*}\Sigma_{k\in V(j)}d_k^+ = D - \Sigma_{j\in J\backslash J^*}\Sigma_{k\in V(j)}d_k^+ \geq D(J^*)$. The last inequality follows from the definitions of D and $D(J^*)$.

Hence $\Sigma_{(i,j)\in C}x'_{ij} \leq K + \Sigma_{j\in J^*}\Sigma_{k\in V(j)}d_k^+$. Since

$$\Sigma_{(i,j)\in C}x_{ij} \leq \Sigma_{(i,j)\in C}x'_{ij} + \Sigma_{(i,j)\in E\backslash Q}x_{ij} + \Sigma_{(i,j)\in(N,V\backslash N)}x_{ij},$$

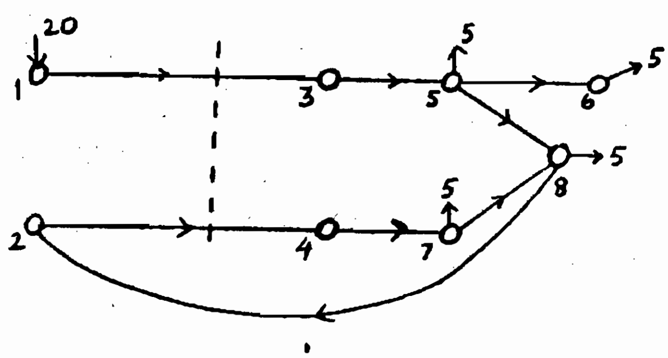
the result follows.

Example 1.

Let $V=\{1,2,3,4,5,6,7,8\}$. If $N=\{3,4,5,6,7,8\}$, $V\backslash N=\{1,2\}$, $C=\{(1,3),(2,4)\}$ and $E=Q=\{(3,5),(5,6),(5,8),(4,7),(7,8)\}$, then

$$x_{13}+x_{24} \leq 10y_{13}+5y_{24}+x_{82}+5$$

is a valid inequality.



Remark 1. Our inequalities tighten the inequalities described by Van Roy and Wolsey (1985). They define $V(j) = j\cup\{k\in N$: there exists a path from j to k using only arcs of Q$\}$. Hence for Example 1, their inequality is $x_{13}+x_{24} \leq 15y_{13}+10y_{24}+x_{82}$. If $y_{13}+y_{24}\geq 1$, our inequality is

tighter. If $y_{13}+y_{24}=0$, then both inequalities are redundant and are implied by the original constraints $x_{ij} \le My_{ij}$.

Remark 2. An obvious way to extend these inequalities is to consider inflow from nodes in $V\backslash N$ to the arcs in C, instead of the outflow into nodes in N. Let $E'=\{(i,j)\in A: i,j\in V\backslash N\}$ and $Q'\subseteq E'$. The basic argument, which is similar to that used in the inequalities described earlier is that flow into arcs in C can come from three sources:

1) arcs in Q'

2) arcs in E'\Q'

3) arcs in $(N, V\backslash N)$.

We omit the details.

4. Relation between Inequalities of different problems.

Cho et. al.(1983) described the following inequalities for the uncapacitated plant location problems. They construct an *intersection graph* for the problem as follows: Let I denote the set of customers and J the set of plants. Then for each customer $i\in I$ they create $|J|$ nodes, one for each plant. If x_{ij} denotes these nodes for $i\in I$ and $j\in J$, then for each i, the nodes x_{ij} for $j=1,...,|J|$ are connected to each other. In addition there is one node y_j for each of the plants. Each node y_j is connected to the nodes x_{ij} for $i=1,...,|I|$. Let N and E denote the node set and edge set respectively. Then for any subset $N^*(s)$ of nodes, we construct the corresponding subgraph G(s) where the node set N(s) and edge set E(s) are defined as follows:

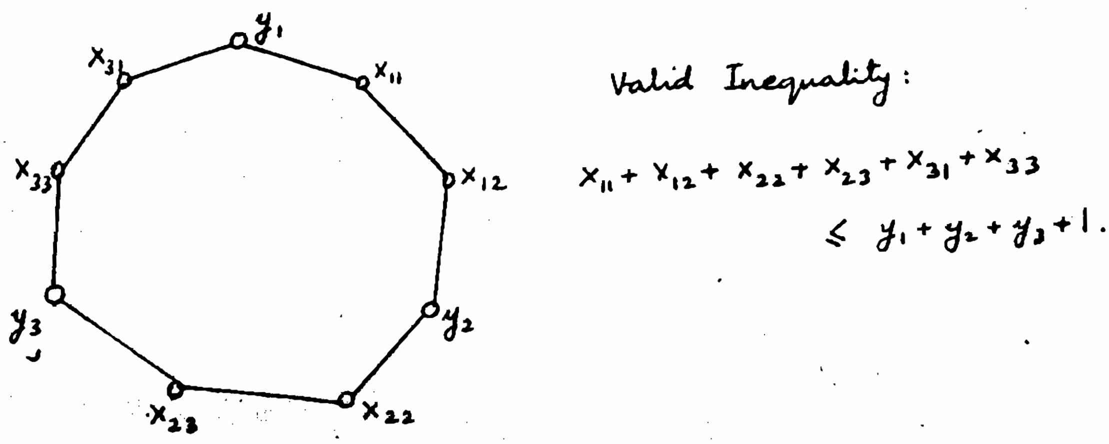
$$N(s)=\{x_{ij}\in N^*(s): x_{ij} \text{ and } y_j \in N^*(s)\}\cup\{y_j\in N^*(s)\}$$

and $E(s)=\{(x_{ij}, x_{ik}): x_{ij}, x_{ik}\in N(s)\}\cup\{(y_j,x_{ij}): y_j, x_{ij}\in N(s)\}$. Let $I(s)=\{i\in I: i\in N(s)\}$, and $J(s)=\{j\in J: j\in N^*(s)\}$.

Then for any connected graph $G(s)$, and $|I(s)|\geq 3$, $|J(s)|\geq 3$, the inequality

$$\Sigma_{(i,j)\in E(s)} x_{ij} \leq \Sigma_{j\in J(s)} y_j + |I(s)| - \beta(G(s))$$

is valid, where $\beta(G(s))$ is the *covering number* of $G(s)$, i.e., the minimum number of plants $j\in J(s)$ necessary to cover all customers $i\in I(s)$.



Let us next consider the single item lot-sizing problem. Barany, Van Roy and Wolsey (1984a) described the convex hull of feasible integer solutions by means of the following valid inequalities. Let y_i denote the integer {0,1} variables indicating whether the machine is setup in period i or not, x_i the production in period i, s_i the stock at the end of period i, $d_i>0$ the demand in period i, and $d_{il}=\Sigma_{t=i}^{l} d_t$ the total demand in periods i through l. Let the planning horizon extend up to period n, and let $L=\{1,2,\ldots,l\}$ for $l\leq n$, and $S\subseteq L$. Then

$$\Sigma_{i\in L\setminus S} x_i + \Sigma_{i\in S} d_{il} y_i \geq d_{1l}$$

is a valid inequality. Using the equation $\Sigma_{i=1}^{l} x_i + s_l = d_{1l}$, we can rewrite the inequality as follows

$$\Sigma_{i\in S} x_i \leq \Sigma_{i\in S} d_{il} y_i + s_l.$$

Consider the lot-sizing problem with an additional start-up cost incurred when the machine is setup in period i, but not in period i-1. Let the integer variables $z_i\in\{0,1\}$ indicate whether we incur a start-up cost or not. Then Sastry (1991) described the following valid inequalities. Let $L = \{1,...,l\}$ with $l \le n$. We partition L into sets X, Y, Z and XZ(t), where $t \le l$, and impose the conditions:

i) if period $i \in X$, then period $i+1 \in X \cup Y$,

ii) if period $i \in YZ(t)$ then periods $i+1$ through $t-1 \in YZ(t)$.

Then,

$$\Sigma_{i\in X} x_i + \Sigma_{i\in Y} d_{il} y_i + \Sigma_{i\in Z} d_{il} z_i + \Sigma_{i\in XZ(t)} (x_i + d_{tl} z_i) \ge d_{1l}$$

is a valid inequality. Using the equation $\Sigma_{i=1}^{l} x_i + s_l = d_{1l}$ we can rewrite the inequality as

$$\Sigma_{i\in Y\cup Z\cup XZ(t)} x_i \le \Sigma_{i\in Y} d_{il} y_i + \Sigma_{i\in Z} d_{il} z_i + \Sigma_{i\in XZ(t)} (x_i + d_{tl} z_i) + s_l .$$

Can we discover any common underlying structure in these inequalities? First, in all cases, there is a set of nodes which are the supply nodes. Thus, in the plant location problem, these are the plants, in the lot-sizing problems these are the dummy node, and in the network design problem, the nodes with $d_i < 0$. Second, we have a set of demand nodes in each case. Thus, some subset of supply nodes sends flow to another subset of demand nodes. The inequalities can be conceptualized as follows: For any cut set of arcs (V\N, N) and any subset of arcs $Q \subseteq \{(i,j): i,j \in N\}$, let F(in) denote the total flow in any subset of arcs $C \subseteq (V\backslash N, N)$, y_a the integer 0,1 variable indicating whether arc a is open or not, c_a the coefficient of y_a, and F(out) the outflow from arcs in C along arcs not in Q. Then all the inequalities have the form

$$F(in) \le \Sigma_{a\in C} c_a y_a + F(out) + constant.$$

The coefficients c_a and the constant are determined as described in theorem 1. Similarly, as indicated in remark 2, for any subset of arcs $Q' \subseteq \{(i,j): i,j \in V\backslash N\}$, let F'(out) denote the total outflow from V\N along any subset of arcs $C \subseteq (V\backslash N, N)$, y_a the integer 0,1 variable indicating whether arc a is open or not, c_a the coefficient of y_a, and F'(in) the inflow from arcs in C along arcs not in Q'. Then we can write the following valid inequalities

$$F'(out) \leq \Sigma_{a\epsilon C} c_a y_a + F'(in) + constant.$$

However, if we use this framework for the special cases of the uncapacitated plant location or the lot-sizing problems, we obtain slightly different inequalities. For instance, in the plant location problem, for any subgraph G(s), we obtain the inequality

$$\Sigma_{(i,j)\epsilon E(s)} x_{ij} \leq |I(s)|$$

if we use the framework developed for the network design problem. But since the plant location problem's underlying network has a special structure, we can tighten these inequalities to obtain the ones described by Cho et.al.(1983).

5. Applications

Some of the practical applications of the general network design problem have already been discussed. In the present context, where information technology is growing rapidly, the demand for greater communication will grow rapidly. Thus we will need to design networks for telephone, facsimile and computer communications. However, much more applied work in the general network design problem needs to be done if we are to benefit from operations research techniques. For large investments in communications of the order of Rs. 10 crores or more, even a 5% saving can result in an overall saving of Rs.50 lakhs. Another area where we can profitably use network design is in planning of regional or national level road transport networks.

From the computational point of view, Lagrangean relaxation techniques have worked well on plant location problems. However, not much work has been done on the general network design problem. Therefore one potential application of the cutting plane methods is to test out the computational efficiency on large scale problems, and compare them with other methods. As mentioned earlier, computational success of this method on large scale travelling salesman problems, 0-1 integer programs, and lot-sizing problems, indicates that it might be worthwhile to study this approach in detail.

6. Summary

We have described a new class of cuts or valid inequalities for the single commodity network design problem, and have shown that they are valid. We also indicate how we can extend this class of valid inequalities. We also develop a general framework for the valid inequalities, and show that in some instances we can generate inequalities for special cases like the lot-sizing and plant location problems.

7. Future directions

Some of the open research questions are: can we further extend the class of valid inequalities described using results from the special cases? For instance, in the network inequalities, can we include nodes $k \in N$ with $d_k < 0$? The capacitated version of the problem, and the multi-commodity versions of the problem also need to be studied. A further problem of interest is to computationally test these inequalities to find out if they are practically useful. Recent research has shown that in many cases valid inequalities can be used to generate *compact* reformulations of integer programs as linear programs, i.e., one where the number of constraints and variables is a polynomial function of the number of variables in the original integer program (see for instance Martin, Rardin and Campbell (1990)). It would be theoretically and computationally useful if we can find alternate compact formulations for the mixed integer network design problem.

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