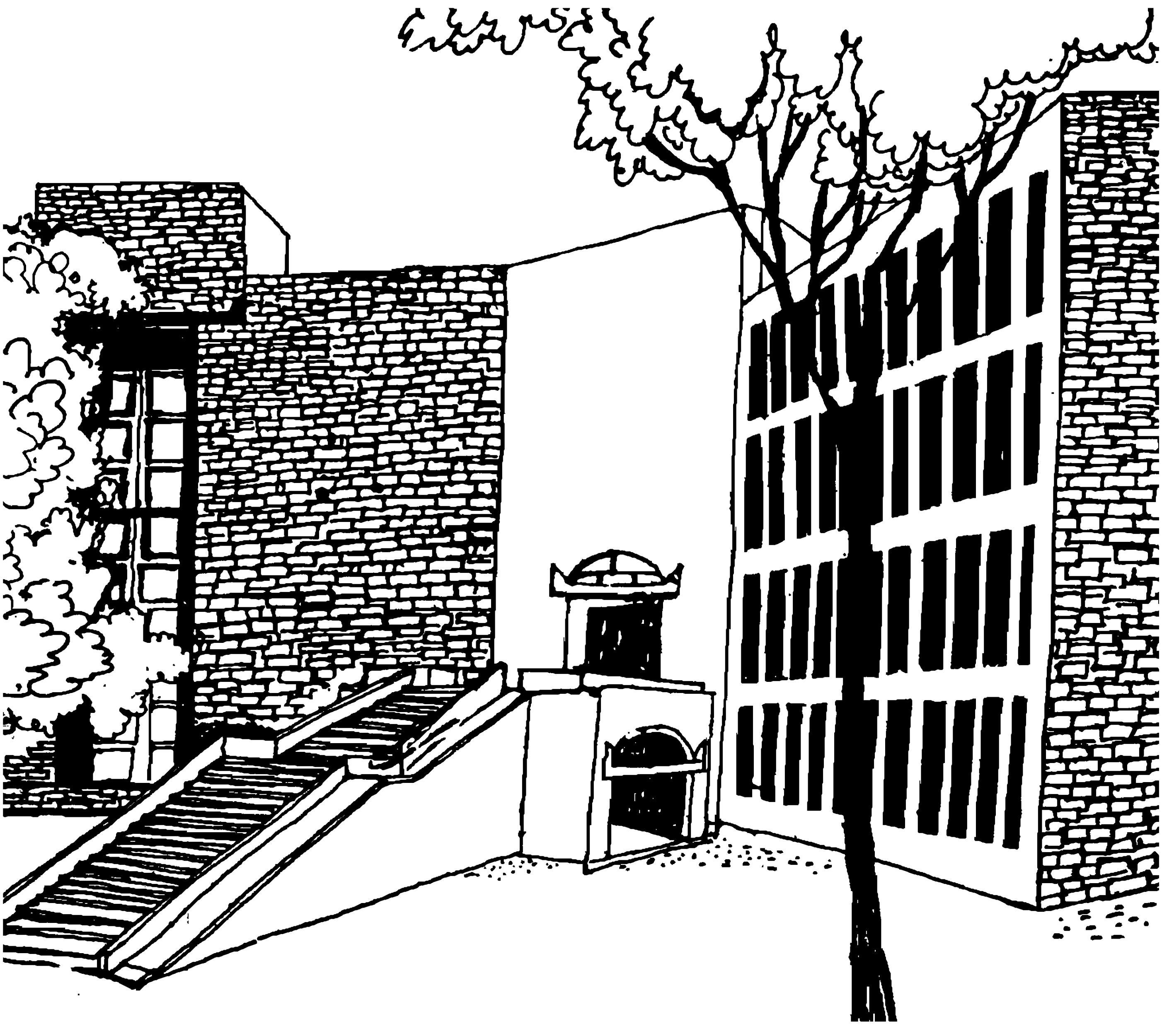
विद्याविनियोगाद्विकासः

IIM

AHMEDABAD

Working Paper



AN AXIOMATIC CHARACTERIZATION OF THE LEXICOGRAPHIC UTILITARIAN COLLECTIVE UTILITY FUNCTION

By

Somdeb Lahiri

W.P. No.1157
December 1993

The main objective of the working paper series of the IIMA is to help faculty members to test out their research findings at the pre-publication stage.

INDIAN INSTITUTE OF MANAGEMENT
AHMEDABAD - 380 015
INDIA

ABSTRACT

In this paper we axiomatically characterize the family of lexicographic utilitarian collective utility functions. As a by-product we obtain the utilitarian collective utility function by imposing a shift anonymity condition. Finally we axiomatically characterize the family of rank k-dictator collective utility functions, as a corollary to our main characterization theorem.

ACKNOWLEDGEMENT

This paper was written when I was visiting CENTRE DE RECHERCHE EN ECONOMIE ET STATISTIQUE (CREST) of the Ecole Nationale de la Statistique et de l'Administration Economique (ENSAE) on a French government scholarship during the months of November and December 1994. I would like to thank my hosts for their kind hospitability and particularly Marc Fleurbaey for stimulating insights as well as for pointing out serious lapses in an earlier version of this paper. However, the usual disclaimer applies.

1. Introduction

A collective utility function is a real valued function defined on the vector of utilities achievable by a society consisting of a finite number of individuals. It is a measure of the welfare embodied in a distribution of utilities from the point of view of a social planner or an autonomous decision maker. A collective utility function is a useful tool in policy analysis. A survey of the theory of collective utility functions can be found in Moulin [1988].

In Kreps [1990], can be found a family of collective utility functions which is characteristically intermediate between two famous utility functions : the generalized utilitarian collective utility function (which considers a weighted sum of the utilities of the individual agents) and the egalitarian collective utility function (which simply considers the utility of the poorest man in the economy). It is formed by first arranging the utilities in ascending order of magnitude and then taking a weighted sum of this arrangement. Unlike in the generalized utilitarian case, the weights are assigned, not to the individuals, but to their ranks on a utility scale. A special case of this family is the egalitarian collective utility function which assigns all the weight to the poorest individual. The (non-generalized) utilitarian collective utility function corresponds to the situation with equal weights for all ranks and the rank-k dictator collective utility function corresponds to the situation with all the weight on the k^{th} rank. We choose to call this family of collective utility functions lexicographic utilitarian.

Our agenda for this paper runs as follows : (i) first we axiomatically characterize the family of lexicographic utilitarian collective utility functions ; as a by-product we obtain the utilitarian collective utility functions by imposing a shift anonymity condition ; (ii) second we axiomatically characterize the family of rank-k dictator collective utility functions, as a corollary to our main characterization theorem.

Earlier characterizations of the utilitarian collective utility function can be found in d'Aspremont and Gevers [1977] and of the rank k-dictator collective utility function in Hammond [1976] and d'Aspremont and Gevers [1977].

2. The framework

As in Moulin [1988], we shall denote by $N = \{1, 2, ..., n\}$ the "society", which is made up of a fixed set of participating agents. A distribution of utilities is a vector $u \epsilon \mathbb{R}^n$, where u_i (the i^{th} coordinate of u) is the utility assigned to the i^{th} agent.

A collective utility function (CUF) is a real-valued function $W : \mathbb{R}^n \rightarrow \mathbb{R}$ satisfying the following property :

$\forall u, v \epsilon \mathbb{R}^n, u \geq v (u >> v) \Rightarrow W(u) \geq W(v) (W(u) > W(v))$.

We refer to the defining property of a collective utility function as unanimity.

We now define the following ranking function $\sigma : \mathbb{R}^n \rightarrow \mathbb{R}^n$:

$\forall u \epsilon \mathbb{R}^n, \{\sigma_1(u), ..., \sigma_n(u)\} \equiv \{u_1, ..., u_n\}$ and $\sigma_1(u) \leq \sigma_2(u) \leq ... \leq \sigma_n(u)$.

A collective utility function is called lexicographic utilitarian, and represented by W^* if there exists n-real numbers $\alpha_1, ..., \alpha_n, \alpha_i \geq 0 \forall i \epsilon N$, $\sum_{i \epsilon N} \alpha_i = 1$, such that

$W^*(u) = \sum_{i \epsilon N} \alpha_i \sigma_i(u)$. $\forall u \epsilon \mathbb{R}^n$.

If $\alpha_1 = 1$, we have the egalitarian collective utility function. In general for $\alpha_k = 1$, $k \epsilon N$, we have the rank-k dictator collective utility function. If $\alpha_k = \frac{1}{n} \forall k \epsilon N$, we have the utilitarian collective utility function.

The following property will turn out to be significant is what follows : (lexicographic zero independence) :

- A collective utility function $W : \mathbb{R}^n \rightarrow \mathbb{R}$ is said to be ***lexicographically zero independent*** if $\forall u, v, w \epsilon \mathbb{R}^n, \lambda > 0$,

$W(u) \geq W(v) \Leftrightarrow W(\lambda \sigma(u) + \sigma(w)) \geq W(\lambda \sigma(v) + \sigma(w))$

Notice the similarity of this property with zero independence the latter having the same definition simply without the $\sigma' sand \lambda$.

3. The main result

We now state and prove the main result of this paper.

Theorem 1 : Let W be a collective utility function which is continuous and satisfies lexicographic zero independence. Then there exists $\alpha_1, ..., \alpha_n \geq 0$ with $\sum_{i=1}^n \alpha_i = 1$ such that $\forall u, v \epsilon \mathbb{R}^n$,

$W(u) \geq W(v) \leftrightarrow \sum_{i=1}^n \alpha_i \sigma_i(u) \geq \sum_{i=1}^n \alpha_i \sigma_i(v)$

Conversely, given $\alpha_1, ..., \alpha_n \geq 0$ with $\sum_{i=1}^n \alpha_i = 1$, the lexicographic utilitarian collective utility function associated with these weights satisfy lexicographic zero independence.

Proof : That a lexicographic utilitarian collective utility function corresponding to a givent set of weights satisfies lexicographic zero independence follows by observing that $W^*(u) = W^*(\sigma(u)) \forall u \epsilon \mathbb{R}^n$.

Conversely let W be a continuous collective utility function satisfying lexicographic zero independence.

Putting $w = 0$ in the definition of lexicographic zero independence we see that W is a monotone increasing transformation of $W_0\sigma$. Let $w \epsilon \mathbb{R}^n$. Define two sets :

$$A = \{\sigma(u)\epsilon\mathbb{R}^n / W(\sigma(u)) > W(\sigma(w))\}, B = \{\sigma(u)\epsilon\mathbb{R}^n / W(\sigma(u)) < W(\sigma(w))\}.$$

Let $\sigma(u^1), \sigma(u^2)\epsilon A$ and let $u^0 = \frac{1}{2}\sigma(u) + \frac{1}{2}\sigma(u^2)$. Observe that $u^0 = \sigma(u^0)$.

Assume $W(\sigma(u^2)) \geq W(\sigma(u^1))$
Then $W(\sigma(u^0) = W(\frac{1}{2}\sigma(u^1) + \frac{1}{2}\sigma(u^2))$
$\geq W(\frac{1}{2}\sigma(u^1) + \frac{1}{2}\sigma(u^1))$
(from lexicographic zero independence).
$= W(\sigma(u^1))$
Thus $\sigma(u^0)\epsilon A$. Further W is continuous.

Thus A is a convex set.

We establish similarly that B is a convex set.

Further $A \bigcap B = \phi, A \neq \phi, B \neq \phi$ (by unanimity).

Thus by a separating hyperplane theorem and unanimity, there exists $\alpha_1, ..., \alpha_n \geq 0$, $\sum_{i=1}^n \alpha = 1$ such that

$\sum_{i=1}^n \alpha_i\sigma_i(u) \geq \sum_{i=1}^n \alpha_i\sigma_i(v) \forall \sigma(u)\epsilon A, \sigma(v)\epsilon B.$

Further $\{\sigma(u)\epsilon\mathbb{R}^n / W(\sigma(u)) = W(\sigma(W))\} \subseteq \{\sigma(u)/u\epsilon\mathbb{R}^n\}(AUB) \equiv C.$

By unanimity C has empty interior.

Thus $\{\sigma(u)\epsilon\mathbb{R}^n / W(\sigma(u)) = W(\sigma(w))\} = \{\sigma(u)\epsilon\mathbb{R}^n / \sum_{i=1}^n \alpha_1\sigma_i(u) = \sum_{i=1}^n \alpha_1\sigma_i(w)\}.$

Hence $W(\sigma(u)) \geq W(\sigma(v)) \leftrightarrow \sum_{i=1}^n \alpha_i\sigma_i \geq \sum_{i=1}^n \alpha_i\sigma_i(v) \forall u, v\epsilon\mathbb{R}^n$.
$\leftrightarrow W_*(u) \geq W_*(v) \forall u, v\epsilon\mathbb{R}^n$.

Since $W_0\sigma$ is monotone increasing transformation of W, we have proved the theorem.

Q.E.D.

Note : In the above proof use was made of the fact that $W(\sigma(u)) \geq W(\sigma(v)) \Rightarrow W(\frac{1}{2}\sigma(u) + \frac{1}{2}\sigma(v)) \geq W(\sigma(v))$

This follows from lexicographic zero independence by taking $\sigma(w) = \sigma(v)$. As a corollary to the above theorem we have the following :

Corollary 1 : Let W be a collective utility function which is continuous, satisfies lexicographic zero independence and the following shift anonymity property : $\forall u \epsilon \mathbb{R}^n$, $\forall \lambda \epsilon \mathbb{R} \forall i, j \epsilon N$,
$W(u_1, ..., u_{i-1}, u_i + \lambda, u_{i+1}, ..., u_n) = W(u_1, ..., u_{j-1}, u_j + \lambda, u_{j+1}, ..., u_n)$

Then $\alpha_k = \frac{1}{n} \forall k \epsilon N$

Proof : Obvious.

Q.E.D.

4. Independence of Common Utility Pace :

- A collective utility function $W : \mathbb{R}^n \rightarrow \mathbb{R}$ is said to be ***independent of common utility pace*** if, for every increasing bijection $f : \mathbb{R} \rightarrow \mathbb{R}$, we have :

$W(u) \geq W(v) \leftrightarrow W(f(u)) \geq W(f(v)) \forall u, v \epsilon \mathbb{R}^n$

Where we denote $f(u) = (f(u_1), f(u_2), ..., f(u_n))$

The above definition can be found in Moulin [1988].

The following theorem which was established independently by Hammond [1976], d'Aspremont and Gevers [1977], now follows as an easy consequence of our theorem 1.

Theorem 2 : The rank-k dictator collective utility function satisfies independence of common utility pace. Conversely if W is a continuous and symmetric collective utility function satisfying independence of common utility pace it must be a monotone increasing transformation of the rank-k dictator collective utility function for some $k \epsilon \{1, ..., n\}$.

Proof : We first observe that independence of common utility pace and symmetry of W implies lexicographic zero independence. Since W is assumed to be continuous, $\exists \alpha_1, ..., \alpha_n \geq 0$, $\sum_{i=1}^n \alpha_i = 1$ such that $\forall u, v \epsilon \mathbb{R}^n$.

$W(u) \geq W(v) \leftrightarrow \sum_{i=1}^n \alpha_i \sigma_i(u) \geq \sum_{i=1}^n \alpha_i \sigma_i(v)$

By independence of common utility pace, for all bijections $f : \mathbb{R} \rightarrow \mathbb{R}$,

$\sum_{i=1}^n \alpha_i \sigma_i(u) \geq \sum_{i=1}^n \alpha_i \sigma_i(v) \leftrightarrow \sum_{i=1}^n \alpha_i f(\sigma_i(u)) \geq \sum_{i=1}^n \alpha_i f(\sigma_i(v)) \forall u, v \epsilon \mathbb{R}^n$.

This implies that there exists $k \epsilon N$ such that $\alpha_i = 0 \, \forall i \neq k$ i.e. W is a monotone increasing transformation of the rank k-dictator solution for some $k \epsilon N$.

Q.E.D.

5. Conclusion : It should be noted that theorem 1 was proved without an explicit symmetry assumption. Lexicographic zero independence implied symmetry of the collective utility function. On the other hand independence of common utility pace above does not not guarantee lexicographic zero independence. However along with symmetry it does.

REFERENCES

1. d'Aspremont, C. and L. Gevers [1977] : "Equity and the informational basis of collective choice", Review of Economic Studies, 44(2), 199-209.

2. Hammond, P. [1976] : "Equity, Arrows Conditions and Rawls difference principle", Econometrica 44(4), 793-804.

3. Kreps, D; [1990] : "A Course in Microeconomic Theory", Prentice Hall.

4. Moulin, H. [1988] : "Axioms of Cooperative Decision Making", Cambridge University Press.