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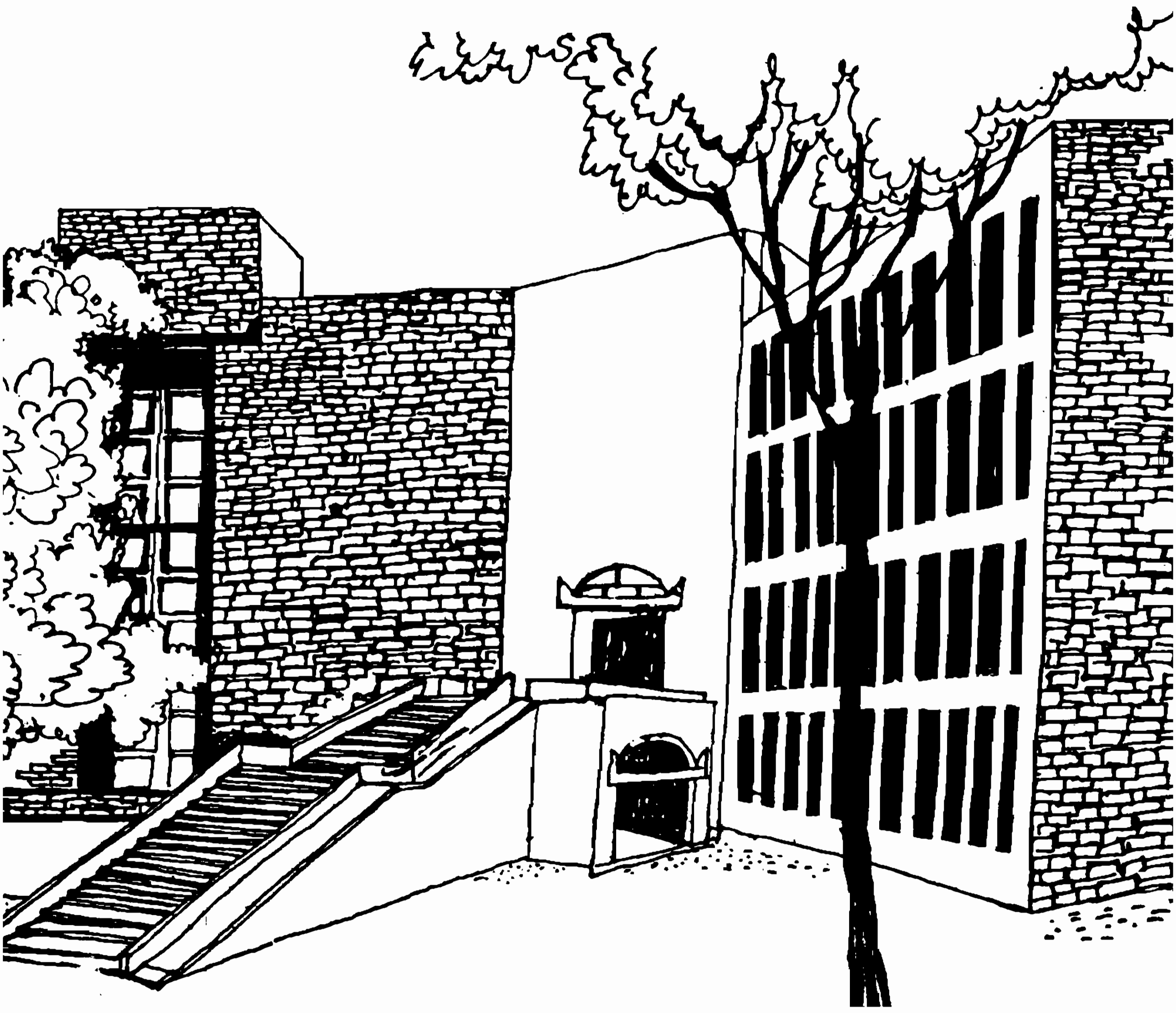

CONSTRAINED EQUILIBRIUM PRICES UNDER FIXED RATIONING OF COMMODITIES

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Abstract

In this paper we prove the existence, restricted efficiency and incremental fairness of constrained equilibrium allocations.

1. Introduction :- In Baumol (1986) can be found the description of a rationing scheme which goes as follows: the rationed commodities are allocated equally among all the consumers; the unrationed commodities are available on the market at market prices. Thus the constraint on the rationed market is such that no individual can purchase more than what equal division of the commodity will allow.

The problem associated with such a scheme is the obvious one: a set of prices have to be found where markets for all goods clear. If prices of the rationed commodities are not sufficiently low, some consumers may end up by purchasing less than what equal division would permit and staking a claim greater than what the market could accommodate for the unrationed commodities. Hence finding a set of constrained equilibrium prices at which markets for all good clear is a problem.

It may be asked, what prompts the choice of equal division as a rationing constraint. The reason behind such a choice is society's perception of what constitutes a fair division of resources. Thus, the problem of decentralizing such an allocation is almost wholly normative. Such methods have to be contrasted, with those concerned with finding suitable rationing constraints in order to clear markets at given fixed prices - a theory which finds adequate expression in Lahiri (1993a,b,c).

In this paper, we prove the existence, restricted efficiency and incremental fairness of the allocations resulting from such rationing schemes.

2. The Model :- We consider a distribution economy consisting of n consumers and l goods. The aggregate availability of the l-goods is summarized in a vector $\omega \in \mathbb{R}^l_{++}$ (the strictly positive orthant of l-dimensional Euclidean space). The income of consumer i($\in${1,...,n}: the index set of consumers) is $w_i > 0$. Let $w = \Sigma^n_{i=1} w_i$ be the total income of the consumers. The preferences of consumer i over alternative consumption vectors is summarized in a utility function $u_i : \mathbb{R}^l_+ \to \mathbb{R}$ where $\mathbb{R}^l_+$ is the non-negative orthant of l-dimensional Euclidean space. For all $i \in \{1,\ldots,n\}$, u_i is assumed to be continuous, strongly increasing (i.e. $x, \bar{x} \in \mathbb{R}^l_+$, $x \geqq \bar{x}, x \neq \bar{x} \Rightarrow u_i(x) > u_i(\bar{x})$) and quasi-concave (ie. $\forall x, \bar{x} \in \mathbb{R}^l_+, \forall t \in [0,1], u_i(tx+(1-t)\bar{x}) \geqq \min(u_i(x), u_i(\bar{x}))$). This in particular implies that $\forall i \in \{1,\ldots,n\}$, u_i is semi-strictly quasi-concave (i.e. $u_i(x) > u_i(\bar{x}) \Rightarrow u_i(tx+(1-t)\bar{x}) > u_i(\bar{x}) \forall t \in (0,1)$ where $x, \bar{x} \in \mathbb{R}^l_+$).

An allocation is an n-tuple $x=(x_i)^n_{i=1}$ such that $x_i \in \mathbb{R}^l_+$ $\forall i \in \{1,\ldots,n\}$. An allocation is said to be feasible if $\Sigma^n_{i=1} x_i \leqq \omega$. A <u>price vector</u> is a vector $p \in \mathbb{R}^l_+$ such that $p.\omega = w$.

Let $S \subseteq \{1,\ldots,l\}$ be the set of rationed commodities. A <u>constrained equilibrium</u> is an allocation-price pair (x,p) such that

(i) $\forall i \in \{1,\ldots,n\}$, $x^j_i = \dfrac{\omega^j}{n}$ $\forall j \in S$ (: super-script denotes the corresponding coordinate of l-dimensional Euclidean space)

(ii) $\Sigma^n_{i=1} x_i \leqq \omega$

(iii) $\forall i \in \{1,\ldots,n\}$, x_i solves:

$\max u_i(y)$

s.t. $p.y \leqq w_i$

$y \geqq 0$, $y^j \leqq \omega^j/n$ $\forall j \in S$

(iv) $p.\omega = w$.

3. Existence of Constrained Equilibrium :- Let

$\Delta = \{p \in \mathbb{R}^l_+ / p.\omega = \Sigma^n_{i=1} w_i \equiv w\}$.

$\forall i \in \{1,\ldots,n\}$, define $d_i : \Delta \rightarrow \mathbb{R}^l_+$ as follows: $x_i \in d_i(p)$ if and only if x_i solves:

$\max u_i(y)$

s.t. $p.y \leqq w_i$

$y \in \mathbb{R}^l_+$, $y^j \leqq \omega^j/n$ $\forall j \in S$, $y^j \leqq \omega^j + 1$ $\forall j \notin S$.

The following lemma has a straightforward proof:

<u>Lemma 1</u> :- If $x_i \in d_i(p)$ and $x^j_i < \omega^j + 1$ $\forall j \notin S$ then x_i solves:

$\max u_i(y)$

s.t. $p.y \leq w_i$

$y \in \mathbf{R}^l_+$, $y^j \leq \omega^j / n$ $\forall j \in S$.

Proof :- Follows easily from the semi-strict quasi-concavity of u_i and the fact that $x^j_i < \omega^j + 1$ $\forall j \in S$.

Q.E.D.

Define $d: \Delta \to \mathbf{R}^l_+$ by $d(p) = \Sigma^n_{i=1} d_i(p)$ d is referred to as the market demand correspondence. It is easy to verify that d is non-empty valued, convex-valued, upper-semi-continuous. Further $y \in d(p) \Rightarrow y \in N \equiv \{z \in \mathbf{R}^l_+ / y^j \leq n\omega^j + n \; \forall j \notin S,\ y^j \leq \omega^j \; \forall j \in S\}$.

We are now ready to prove our main theorem:

Theorem 1 :- Under the assumptions of this paper, a constrained equilibrium exists for the economy under consideration.

Proof :- Let $\Delta(S^c) = \{p \in \Delta / p^j = 0 \; \forall j \in S\}$.

Consider the following function $f: \Delta(S^c) \times N \to \Delta(S^c)$:

$$f^j(p,z) = \frac{p^j + \max(0, z^j - \omega^j)}{1 + \Sigma_{j \notin S} [\max(0, z^j - \omega^j)] \frac{\omega^j}{w}} \quad \text{for } j \notin S$$

$$= 0 \quad \text{for } j \in S$$

Consider the following correspondence $g: \Delta(S^c) \times N \twoheadrightarrow N$:

$g(p,z) = d(p)$

By strong monotonicity of preferences, $y \in d(p) \Rightarrow y^j = \omega^j \; \forall j \in S$.

The correspondence $f \times g$: $\Delta(S^c) \times N \to \Delta(S^c) \times N$ is non-empty valued, convex valued, upper semi-continuous. Hence by Kakutani's fixed point theorem, there exists $(p^*, x^*) \in \Delta(S^c) \times N$ such that $(p^*, x^*) \in f(p^*, x^*) \times g(p^*, x^*)$. By Lemma 1, it is easy to verify that (x^*, p^*) is a constrained equilibrium.

Q.E.D.

We have thus proved the existence of a constrained equilibrium, and in particular one where the rationed commodities are offered free. Thus the resulting pricing scheme will have a inflationary impact on the non-rationed commodities, compared to one where all goods are available on the open market. Thus welfare programs are inherently inflationary - a cost we must bear in mind in deciding which goods should be rationed. The above proof closely resembles the proof of the existence of a market equilibrium Lahiri (1993a).

4. Restricted Pareto Efficiency of Constrained Market Equilibria :- As pointed out by Baumol (1986), the issue here is not efficiency but distributive justice. Thus it should come as no surprise that in general constrained market equilibria are not Pareto efficient.

Example :- Consider a two person two good economy i.e. n=2, l=2, $\mathbf{w}=(2,2)$, $u_1(x,y)=x^{1/2}y$, $u_2(x,y)=xy^{1/2}$. Let the first good be rationed i.e. each person gets one unit of the first good. If $w_1 = w_2 = 2$ then each person getting 1 unit of the second good as well results in a constrained equilibrium with the price of the second good being 2. However an allocation where agent 1 gets 1/2 units of the first good and 3/2 units of the second and agent 2 gets 3/2 units of the first good and 1/2 units of the second is strictly better for both agents. Hence the constrained market equilibrium is not Pareto efficient.

Thus we define the following: An allocation $x \in (\mathbb{R}^l_+)^n$ is said to be restricted Pareto efficient if

(i) $x^j_i = \mathbf{w}^j / n \ \forall j \in S$, $\Sigma^n_{i=1} x^j_i = \mathbf{w}^j \ \forall j \notin S$.

(ii) there does not exist $y \in (\mathbb{R}^l_+)^n$ with $y^j_i = \frac{\mathbf{w}^j}{n} \ \forall j \in S$, $\Sigma^n_{i=1} y^j_i = \mathbf{w}^j \ \forall j \notin S$ with $u_i(y_i) \geq u_i(x_i) \ \forall i=1,\ldots,n$ and one strict inequality.

We then have the following result:

Theorem 2 :- Let x^* be a constrained equilibrium allocation. Then x^* satisfies restricted Pareto efficiency.

Proof :- Let p^* be the corresponding constrained equilibrium price vector. If $u_i(y_i) > u_i(x^*_i)$ then $p^*.y_i > w_i = p^*.x^*_i$.

Let $u_i(y_i) = u_i(x^*_i)$ and $p^*.y_i \leq p^*.x^*_i$. Let $z^j_i = \frac{\omega^j}{n}$ $\forall j \in S$ and $z^j_i > y_i$ such that $p^*.z_i < p^*.x^*_i$. Such a z_i can always be chosen. By strong monotonicity $u_i(z_i) > u_i(x^*_i)$, contradicting that x^*_i solves agent i's constraint maximization problem. Thus $u_i(y_i) = u_i(x^*_i) \Rightarrow p^*.y_i \geq p^*.x^*_i$.

So, $w = \Sigma^n_{i=1} p^*.x^*_i > \Sigma^n_{i=1} p^*.y_i = p^*.\omega = w$, for some $y \in (\mathbb{R}^l_+)^n$ if x^* is not restricted Pareto efficient. This is a contradiction. Hence x^* is restricted Pareto efficient.

Q.E.D.

The converse of the above theorem is a straightforward application of the separating hyperplane theorem (Nikaido (1967)).

Theorem 3 :-Let x^* be a restricted Pareto efficient allocation. Then $\exists t \in \mathbb{R}^*$ such that $t_i \geq -w_i$ $\forall i \in \{1,\dots,n\}$, $\Sigma^n_{i=1} t_i = 0$ and a price vector p^* such that (x^*, p^*) is a constrained equilibrium for the economy described in Section 2 with the income of agent i being $w_i + t_i$.

Proof :- Let $U = \{(\Sigma^n_{i=1} x^j_i)_{j \in S^c} \in \mathbb{R}^{S^c}_+ / u_i((\frac{\omega^j}{n})_{j \in S}, x_i) > u_i(x^*_i)\}$.

U is non-empty (since each u_i is strongly monotonic), convex (since each u_i is semi-strictly quasi-concave), and $(\Sigma^n_{i=1} x^{*j}_i)_{j \in S^c}$ belongs to the boundary of U. Thus there exists $p^* \in \mathbb{R}^{S^c} \setminus \{0\}$ such that $p^*.z > p^*[(\Sigma^n_{i=1} x^{*j}_i)_{j \in S^c}]$ $\forall z \in U$.

It is now easy to check that (x^*, p^*) is a constrained equilibrium for $t_i = p^*.x_i - w_i$, $i = 1,\dots,n$.

Q.E.D.

Thus in some respects a constrained equilibrium has properties resembling a market equilibrium. As shown in Blad and Keiding (1990), a market equilibrium is Pareto efficient and any Pareto efficient allocation can be retrieved as a market equilibrium after suitable redistribution of income.

5. Issues in Fairness :- Ultimately the reason behind a rationing scheme is not efficiency but perceived fairness. This is precisely the reason why Baumol (1986) invokes the particular rationing scheme characterized by a constrained equilibrium.

Let x^* and x be two allocations with $x^{*j}_i = \bar{x}^j_i \forall j \in S$ and $i \in \{1,...,n\}$. x^* is said to be partially incrementally fair with respect to $\bar{x}$ if $u_i(x^*_i) \geq u_i(\bar{x}_i + (x^*_k - \bar{x}_k)) \forall i,k \in \{1,...,n\}$. Under the same hypothesis, x^* is said to be partially superequal with respect to $\bar{x}$ if $u_i(x^*_i) \geq u_i(\bar{x}_i) \forall i=1,...,n$.

Theorem 4 (Baumol) :- Let (x^*, p^*) be a constrained equilibrium and $\bar{x}$ be an allocation such that $\bar{x}^j_i = \dfrac{\omega^j}{n}$ $\forall j \in S$, $i \in \{1,...,n\}$. If $p^*.\bar{x}_i \leq w_i \forall i \in \{1,...,n\}$, then x^* is partially super-equal with respect to $\bar{x}$. In particular $\bar{x}$ could be the allocation where $\bar{x}^j_i$ is the market equilibrium consumption of consumer i for $\forall j \notin S$.

proof :- The proof follows from the observation that $\bar{x}_i$ satisfies the constraints of the ith consumer's maximization problem in the definition of a constrained equilibrium.

Q.E.D.

Baumol (1986) shows that if x^* is partially superequal with respect to $\bar{x}$, then x^* is partially incrementally fair with respect to $\bar{x}$, although the converse need not be true. Thus given the consumptions on the rationed market, no agent envies the trades that any other agent makes on the unrationed market.

Of course in one obvious sense a constrained equilibrium is necessarily fair:

Let $R \subseteq \{1,\ldots,l\}$, $R \neq \emptyset$.

We say that an allocation x^* is partially fair with respect to R if $\forall i \in \{1,\ldots,n\}, u_i(x^*_i) \geq u_i(y_{ik}) \forall k \in \{1,\ldots,n\}$ where $\forall i,k \in \{1,\ldots,\}, y_{ik} \in \mathbb{R}^l_+$, with $y^j_{ik} = x^{*j}_k \forall j \in R$ and $y^j_{ik} = x^{*j}_i \forall j \in R^c$.

In our case since $x^{*j}_i = \frac{\omega^j}{n} \forall i=1,\ldots,n, \forall j \in S$, a constrained equilibrium allocation is partially fair with respect to S. Hence its appeal as a rationing procedure.

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