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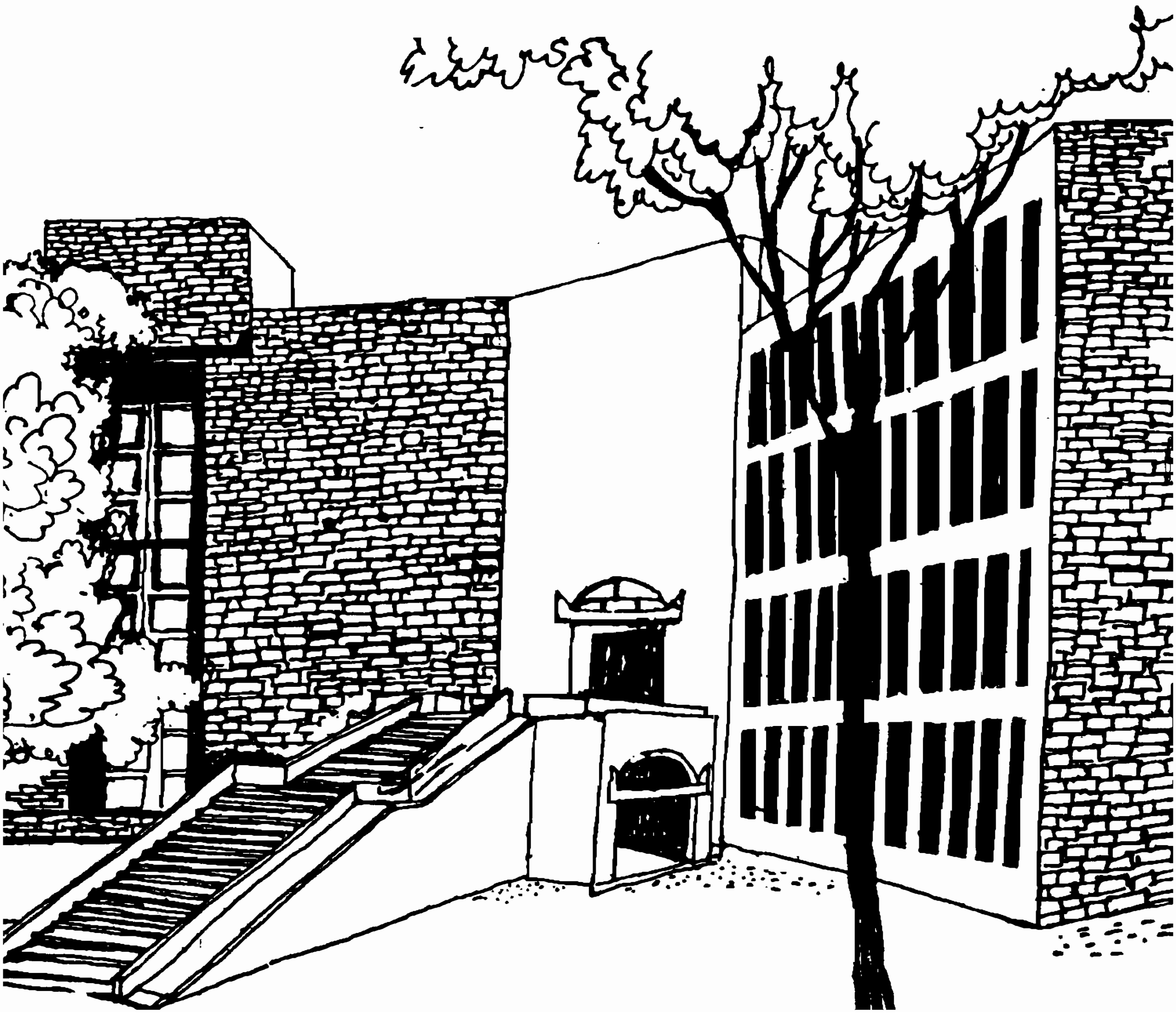

AN AXIOMATIC CHARACTERIZATION OF THE CONSTRAINED EQUAL AWARDS SOLUTION FOR RATIONING PROBLEMS

By

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Abstract

In this paper we propose two different, yet related axiomatic characterizations of the Constrained Equal Awards Solution for rationing problems. The solution itself and its implications are studied in the context of an item in the common minimum programme of the United Front government (which assumed office on June 1st 1996) viz. its decision to supply essential commodities to consumers in the lower income group at half the market price.

1. Introduction: In many real world economic situations we are confronted with the problem of excess demand and thus the necessity to ration the supplies among several customers. This is observed most often, when the government imposes a price ceiling on a commodity which is lower than the equilibrium price i.e. the price at which demand equals supply. Silvestre (1986) contains an interesting discussion of such price rigities resulting in excess demand and its myriad possibilities. More recently, the United Front government which assumed office in India on June 1st 1996, announced a common minimum programme which was jointly endorsed by the fourteen constituent parties comprising the front. One of the items on the agenda of the common minimum programme is that essential commodities would be supplied to consumers in the lower income group, through fair price shops, at half the market price. Assuming a downward sloping demand curve and fixed supplies (i.e. a vertical supply curve), if market price stands for the market equilibrium price, this would of necessity lead to a situation of excess demand. However, the consumers would be willing to pay for the given supply, a much higher price than the administered one. Consequently, there is a welfare increase for the low income consumers visualised by this policy.

In order to implement such a scheme where there is excess demand, rationing rules will have to be adopted. Not all consumers will have the same demand; some will demand more, some less. How should one go about allocating the supplies amongst this disparate group of consumers?

Surprisingly this problem has interesting precedents in the Babylonian Talmud under the guise of the bankruptcy problem: a man dies leaving behind an estate which has to be divided between a group of creditors; further the total debt of the deceased exceeds the estate. This problem was addressed by ancient Jewish scholars.

In recent years fresh impetus has been imparted to the study of this problem by O'Neill (1982), Aumann and Maschler (1985) and Curiel, Maschler and Tijs (1988), who adopt game theoretic methods to resolve this problem. One of the best known rules is the Constrained Equal Awards (CEA) rule which in our demand-supply context would approximately go as follows: the low demanders get what they want; the high demanders all get an equal amount, which nevertheless is not less than what the low demanders get individually. Recent axiomatic characterizations of this rule can be found in Dagan (1996) and Lahiri (1996). Analytically the last two papers differ in that Dagan's characterization is embedded in a fixed population framework, whereas the framework of the latter paper is one of variable population. Considering the popularity of this rule in the literature on rationing (see for instance Benassy (1982)), in this paper we propose a simple axiomatic characterization of the fixed population variety. In an appendix to this paper we provide an alternative characterization of the CEA solution by considerably relaxing one property and slightly strengthening another. It is hoped that these results will enhance the acceptability of the CEA solution among policy makers and administrative personnel.

2. The Model: We consider a set of n agents where $n \in \mathbb{N}$, the set of natural numbers. Let N= {1,2,...,n} be the agent set. A rationing problem is an ordered pair

$(d,S) \in \mathbb{R}^n_+ x \mathbb{R}_+$ such that $\sum_{i=1}^{n} d_i > S$. Let B denote

the set of all rationing problems. Given $(d,S) \in B, d_i$ is the demand of the i^{th} agent and S is the supply. That

$D(d) = \sum_{i=1}^{n} d_i > S,$ emphasizes excess demand.

Given $(d,S) \in B$ an allocation for (d,S) is a vector $x \in \mathbb{R}^n_+$ such that

(i) $x_i \le d_i \; \forall i=1,...,n$

(ii) $\sum_{i=1}^{n} x_i = S$

A solution is a function $F: B \to \mathbb{R}^n_+$ such that F (d,S) is an allocation for (d,S) whenever $(d,S) \in B$.

The Constrained Equal Awards solution $CEA: B \to \mathbb{R}^n_+$ is defined as follows: $CEA(d,S) = x \text{ where } x_i = \min(\lambda, c_i), i \in N$

and $\sum_{i=1}^{n} x_i = S$.

It is well known that for each $(d,S)\epsilon B$, a unique $\lambda\geq 0$ exists which defines CEA(d,S).

2. Properties:- We now state two properties which the constrained equal award solution satisfies.

Equal Treatment (ET):- Given $(d,S)\epsilon B, d_i=d_j\rightarrow F_j(d,S)=F_i(d,S)$.

Equal Treatment is standard and simple. It says, if two people make the same demands then they get identical awards. As a postulate of impartiality, nothing could be more meaningfull.

Insensitivity to Irrelevant Inflations (III):- Given $(d,S),(d',S)\epsilon B$ if $d_i=d_i'\forall i\neq k, d_k\leq d_k'$ and $F_k(d,S)<d_k$ *then* $F(d,S)=F(d',S)$

Insensitivity to Irrelevant Inflations is a vieled strategy proofness type of condition which says that unilateral deviations do not affect outcomes, provided one's demand is not met originally. It is not as mild a property as equal treatment; yet it provides the required force to characterize the CEA solution. It should be noted, that the solution is insensitive to inflation of demand by an individual, if the award for the individual was originally less than what was originally demanded. This is the gist of the III property.

3. Main Theorem:- We now state and prove the main theorem of this paper.

Theorem A:- The only solution to satisfy ET and III is CEA.

Proof:- That CEA satisfies these two properties is quite easy to see. Hence suppose F is a solution which satisfies these two properties and towards a contradiction assume $F \neq CEA$. Thus there exists $(d,S) \in B$ such that $F(d,S) \neq CEA(d,S)$ Without loss of generality and only to make the proof easier assume $d_i \leq d_{i+1}\ \forall i=1,\ldots,n-1$. Clearly there exists $i,j \in N, i<j$ such that

$F_i(d,S) < d_i, F_j(d,S) \leq d_j$ and $F_i(d,S) \neq F_j(d,S)$.

Define $d' \in \mathbb{R}^n_+$ as follows:

$$d'_k = d_k\ \forall k \neq i$$

$$d'_i = d_j.$$

By III property, $F(d',S) = F(d,S)$

By ET property, $F_i(d',S) = F_j(d',S)$

$$\therefore F_i(d,S) = F_j(d',S) = F_j(d,S)$$

This contradiction establishes the theorem.

Q.E.D.

4. Conclusion:- What is the implication of Theorem A for the kind of resource allocation problem envisaged in the common minimum programme, discussed in the introduction?

Basically, Theorem A would say that the CEA solution is the unique solution which is impartial and non-manipulable by demanders whose demands have not been met. This is probably why this solution has such a strong appeal.

The basic thrust of the CEA solution to the problem we discussed in the introduction would be the following: since even among low income consumers there are the high demanders and low demanders of essential commodities, depending largely on family size, one would need to choose a cut off family size (say a family consisting of 5 members). Families of size smaller than this cut of size would get there individual demands depending on size; families of size greater than or equal to the cut of size will get only what a family of the cut of size gets (i.e. in the latter situation allotment takes place per family whereas in the former situation allotment takes place per individual). As a by-product, such a scheme would induce voluntary adoption of family planning measures by low income households: an objective whose importance cannot be overemphazized in the Indian context.

Appendix

The Independence of Irrelevant Inflation property invoked in the paper can be objected to on grounds that it requires insensitivity of the solution (entire vector) to unilateral deviations. What may be more reasonable and less demanding is the following:

Weak Independence of Irrelevant Inflation (WI[3]):-

Given (d,S), $(d',S) \in B$ if $d_i = d'_i$ $\forall i \neq k$, $d_k \leq d'_k$ and

$$F_k(d,S) < d_k \qquad \text{then} \qquad F_k(d,S) = F_k(d',S)$$

In (WI[3]) we reserve our previous comments on insensitivity only for the deviating individual. (WI[3]) along with (ET) does not appear to characterize the CEA solution uniquely. If we strengthen (ET) slightly to a Weak Monotonicity (WM) property, then (WI[3]) along with (WM) uniquely characterizes the CEA solution.

Weak Monotonicity (WM):- Given $(d,S) \in B$ if $d_i \leq d_j$ then $F_i(d,S) \leq F_j(d,S)$.

This property says that higher demanders do not get lesser amounts. It is easy to see that Weak Monotonicity implies Equal Treatment, though not conversely

Theorem B:- The only solution to satisfy WM and WI^3 is CEA.

Proof:- It is easy to see that CEA satisfies these two properties. Hence suppose F is a solution which satisfies these two properties and towards a contradiction assume $F \neq CEA$. Thus there exists $(d,S) \in B$ such that $F(d,S) \neq CEA(d,S)$ Without loss of generality and in order to facilitate the proof assume $d_k \leq d_{k+1}$ $\forall k=1,\ldots,n-1$. Clearly there exists $i,j \in N, i<j$ such that $F_i(d,S) < d_i, F_j(d,S) \leq d_j$ and $F_i(d,S) \neq F_j(d,S)$. By $WM, F_i(d,S) < F_j(d,S)$. By WM once again we may assume, j = n and $i = \min\{k / F_k(d,S) < d_k\}$ By $WM, F_i(d,S) < F_n(d,S)$.

Define $d' \in \mathbb{R}^n_+$ as follows:

$$d'_k = d_k \ \forall k \neq i$$

$$d'_i = d_n$$

By $WI^3, F_i(d',S) = F_i(d,S)$

By ET (which is implied by WM), $F_n(d',S) = F_i(d',S)$.

Thus $F_n(d',S) = F_i(d,S) < F_n(d,S)$.

Clearly there exists k such that i<k<n and $F_k(d',S) > F_k(d,S)$.

But k>i implies by $WM, F_k(d,S) \geq F_i(d,S) = F_n(d',S)$.

Thus $F_k(d',S) > F_n(d',S)$ which contradicts WM since k<n.

Q.E.D.

However for n = 2, (WI^3) and (ET) uniquely characterizes the constrained equal award solution, as the following (which is a considerable strengthening of the previous theorem) reveals.

Theorem: For n = 2, the only solution to satisfy (WI^3) and ET is CEA.

Proof:- Suppose towards a contradiction, that there exists a rationing problem $(d_1, d_2; S)$ and a solution f satisfying (WI^3) and (ET) such that $f(d_1,d_2;S) \neq CEA(d_1,d_2;S)$. Let $(x_1,x_2)=f(d_1,d_2;S)$. Thus $x_1 \neq x_2$ *and* $x_1 < d_1$ where we have assumed without loss of generality $d_1 \leq d_2$. By ET, we must have $d_1 < d_2$. Let $d_1' = d_2$.

By (WI^3), $f_1(d_1', d_2; S) = x_1$.

By ET, $f_2(d_1', d_2; S) = x_1$.

$\therefore 2x_1 = S = x_1 + x_2$, *contradicting* $x_1 \neq x_2$.

This proves the theorem.

Q.E.D.

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