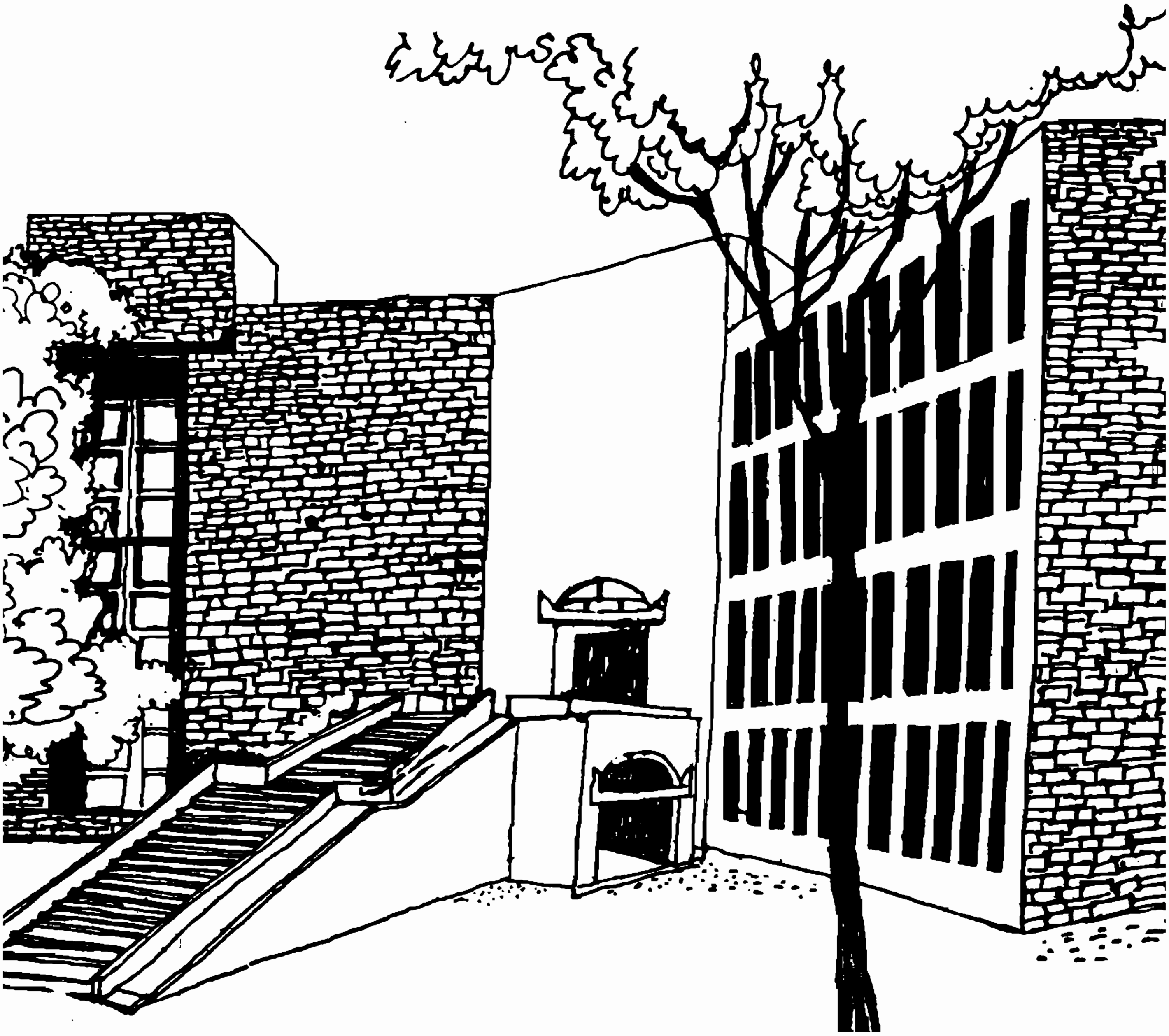
Working Paper



CHARACTERIZING THE EQUAL INCOME MARKET EQUILIBRIUM CHOICE CORRESPONDENCE: AVERAGE ENVY FREENESS INSTEAD OF INDIVIDUAL RATIONALITY FROM EQUAL DIVISION

By

Somdeb Lahiri

W.P. No.1417
December 1997

The main objective of the working paper series of the IIMA is to help faculty members to test out their research findings at the pre-publication stage.

INDIAN INSTITUTE OF MANAGEMENT
AHMEDABAD - 380 015
INDIA

Abstract

In problems of fair division of a given bundle of resources, amongst a finite number of agents, individual rationality from equal division plays a significant role. In a society, where all resources are socially owned, one cannot argue in terms of equal ownership of the social endowment. One normally takes the position that each agent has the right to veto any allocation, which leaves him/her worse than equal division. Based on this premise, individual rationality from equal division has been proposed as a minimal requirement of distributive justice.

In Thomson (1982), we find an equity criterion called average envy-freeness, which in the context of economies with convex preferences, implies individual rationality from equal division. Average envy-freeness says that no agent finds the average consumption of the other agents, superior to his/her own consumption. This concept has been developed in the Foley (1967) tradition of an envy-free allocation: no agent should find his/her consumption inferior to the consumption of any other agent. We show in this paper (the not too difficult result) that average envy-freeness does not automatically imply individual rationality from equal division.

A solution concept which recurs with seeming regularity in the literature of fair division is the equal income market equilibrium solution concept. In a variable population framework Thomson (1988) provides an axiomatic characterization, using the axiom of consistency. Consistency basically says that the departure of some agents with their allocated consumption, should not affect the consumption of the remaining agents, provided they operate the same distribution mechanism as before. Lahiri (1997a, 1997b) use this same axiom to characterize the equal income market equilibrium choice correspondence in convex and non-convex environments. Our main result reported in this paper is similar to a Lahiri (1997b) result, although it may not extend to the non-convex economies considered there. It is thus a modest generalization of the Thomson (1988) result. We use consistency, replication invariance, efficiency and average envy-freeness to show that if a solution satisfies these properties, it must consist of equal income market equilibrium allocations. Subsequently we drop consistency and arrive at yet another characterization of subsolutions of equal income market equilibrium choice correspondence using the strict envy-freeness property due to Zhou (1992).

1. Introduction:- In problems of fair division of a given bundle of resources, amongst a finite number of agents, individual rationality from equal division plays a significant role. In a society, where all resources are socially owned, one cannot argue in terms of equal ownership of the social endowment. One normally takes the position that each agent has the right to veto any allocation, which leaves him/her worse than equal division. Based on this premise, individual rationality from equal division has been proposed as a minimal requirement of distributive justice.

In Thomson (1982), we find an equity criterion called average envy-freeness, which in the context of economies with convex preferences, implies individual rationality from equal division. Average envy-freeness says that no agent finds the average consumption of the other agents, superior to his/her own consumption. This concept has been developed in the Foley (1967) tradition of an envy-free allocation: no agent should find his/her consumption inferior to the consumption of any other agent. We show in this paper (the not too difficult result) that average envy-freeness does not automatically imply individual rationality from equal division.

A solution concept which recurs with seeming regularity in the literature of fair division is the equal income market equilibrium solution concept. In a variable population framework Thomson (1988) provides an axiomatic characterization, using the axiom of consistency. Consistency basically says that the departure of some agents with their allocated consumption, should not affect the consumption of the remaining agents, provided they operate the same distribution mechanism as before. Lahiri (1997a, 1997b) use this same axiom to characterize the equal income market equilibrium choice correspondence in convex and non-convex environments. Our main result reported in this paper is similar to a Lahiri (1997b) result, although it may not extend to the non-convex economies considered there. It is thus a modest generalization of the Thomson (1988) result. We use consistency, replication invariance, efficiency and average envy-freeness to show that if a solution satisfies these properties, it must consist of equal income market equilibrium allocations. Subsequently we drop consistency and arrive at yet another characterization of subsolutions of equal income market equilibrium choice correspondence using the strict envy-freeness property due to Zhou (1992).

2. <u>The Model</u>:- Let $\mathbb{R}$ denote the real line, $\mathbb{R}_+$ the set of non-negative reals and $\mathbb{R}_{++}$ the set of strictly positive reals. Let $\mathbb{N}$ denote the set of all strictly positive integers. Given $\phi \neq X \subset \mathbb{R}$ and Q any non-empty finite set, let X^Q denote the set of all functions from Q to X.

Let P be the collection of all non-empty finite sets Q, where to avoid Russell's paradox we assume that the statements "Q belongs to Q" and "Q does not belong to Q" have no meaning. Given $Q \in P$ and $k \in \mathbb{N}$, let Q^k be the set Q x {1, ..., k}. Clearly $Q^k \in P$.

Let there be $L \geq 2$, $(L \in \mathbb{N})$ infinitely divisible goods in the economy. The commodity space is $\mathbb{R}^L$ and the consumption set of any conceivable agent (consumer) is $\mathbb{R}^L_+$.

Any $Q(\in P)$, is an agent set.

An economy E is a pair $\langle (u^i)_{i\in Q}, \omega \rangle$ satisfying the following properties:-

(i) $Q \in P$, is the agent set

(ii) $\omega \in \mathbb{R}^L_{++}$, is the aggregate social endowment

(iii) $\forall\, i\in Q,\ u^i : \mathbb{R}^L_+ \to \mathbb{R}$ is a continuous and weakly increasing (i.e. $x^i, y^i \in \mathbb{R}^L_+, x^i \gg y^i$ implies $u^i(x^i) > u^i(y^i)$) utility function, which is continuously differentiable in $\mathbb{R}^L_{++}$.

(iv) $\forall\, i\in Q,\ x^i, y^i \in \mathbb{R}^L_+, x^i \in \mathbb{R}^L_{++}$ and $u^i(x^i) = u^i(y^i)$ implies $y^i \in \mathbb{R}^L_{++}$.

(v) $\forall\, i \in Q,\ u^i : \mathbf{R}_+^L \to \mathbf{R}$ is semi-strictly quasi-concave

(i.e.

$x^i,\ y^i \in \mathbf{R}_+^L,\ t \in (0,1),\ u^i(x^i) > u^i(y^i)$ implies

$u^i(tx^i + (1-t)y^i) > u^i(y^i)$).

Let $\mathscr{E}$ be the set of all economies satisfying the above properties.

Given $E = \langle (u^i)_{i\in Q},\ \omega \rangle$, let

$A(E) = \{(x^i)_{i\in Q} \in \mathbf{R}_+^Q \ /\ \Sigma_{i\in Q} x^i = \omega\}$.

A (E) is the set of all feasible allocations for E.

Given E as above a feasible allocation $(x^i)_{i\in Q}$ is said to be Pareto Optimal, if there does not exist $(y^i)_{i\in Q} \in A(E)$ such that $u^i(y^i) > u^i(x^i)\ \forall i \in Q$. [This is actually the definition of Weak Pareto Optimality; however it is easy to show that for our kind of economies Weak Pareto Optimal allocations and Pareto Optimal allocations coincide.] Let P (E) denote the set of all Pareto Optimal allocations. It is easy to show that $P(E) \neq \phi$

Given E as above let

$IR(E) = \left\{(x^i)_{i\in Q} \in A(E) \,/\, u^i(x^i) \geq u^i\left(\frac{\omega}{|Q|}\right) \forall\, i\in Q\right\}$. IR(E) is the set of all allocations for E which are Individually rational from equal division.

Given E as above and $k\in\mathbb{N}$, E^k denotes the k-replica of E where E^k is defined as follows:

$$E^k = \langle (u^{i,j})_{(i,j)\in Q^k},\ k\omega \rangle \quad \text{where} \quad \forall (i,j) \in Q^k, u^{i,j} = u^i.$$

Given $E\in\mathscr{E}$ and $(x^i)_{i\in Q} \in A(E)$ (where Q is the agent set for E) and $k \in \mathbf{N}$, let $(y^{(i,j)})_{(i,j) \in Q^k}$ be defined as

$y^{(i,j)} = x^i \forall (i,j) \in Q^k$.

Given $E \in \mathscr{E}$, a feasible allocation $(\overline{x}^i)_{i \in Q}$ is said to be a price equilibrium if there exists $p \in \mathbf{R}^L_+ \setminus \{0\}$ such that $\forall i \in Q$, $\overline{x}^i$ solves

$$u^i(x^i) \to \max$$

$$s.t.\ p.x^i \leq p.\overline{x}^i$$

$$x^i \in \mathbb{R}^L_+$$

The following result is standard.

<u>Proposition 1</u>:- Given $E \in \mathscr{E}$

(i) every price equilibrium allocation is Pareto Optimal.

(ii) every Pareto Optimal allocation is a price equilibrium allocation, with respect to a unique price $p \in \mathbb{R}^L_+ \setminus \{0\}$

Let $V \subset \mathscr{E}$ be given

A choice correspondence on V is a non-empty valued correspondence $F: V \rightarrow\rightarrow \bigcup_{Q \in P} (\mathbb{R}^L_+)^Q$ such that $\forall E \in V, F(E) \subset A(E)$.

A choice correspondence F on V is said to be efficient if $F(E) \subset P(E)\ \forall E \in V$.

A choice correspondence F on V is said to be individually rational (from equal division) if $F(E) \subset IR(E)\ \forall E \in V$.

A choice correspondence F on V is said to be consistent if $\forall E = \langle (u^i)_{i \in Q}, \omega \rangle \in V, \forall \phi \neq M \subset Q$,

$\forall (x^i)_{i\in Q} \in F(E), E' = \langle (u^i)_{i\in M}, \Sigma_{i\in M} x^i \rangle \in V$ implies

$(x^i)_{i\in M} \in F(E')$.

A choice correspondence F on V is said to be replication invariant if

$\forall E = \langle (u^i)_{i\in Q}, \omega \rangle \in V, \quad \forall k \in \mathbb{N}, \ (x^i)_{i\in Q} \in F(E)$ implies

$(y^{(i,j)})_{(i,j)\in Q^k} \in F(E^k)$ provided $E^k \in V$.

A choice correspondence F on V is said to be average envy-free if

$\forall E = \langle (u^i)_{i\in Q}, \omega \rangle \in V, \ \forall \ i \in Q, S = Q \setminus \{i\}$ and $(x^i)_{i\in Q} \in F(E)$, we

have $u^i(x^i) \geq u^i\left(\frac{1}{|S|} \sum_{j\in S} x^j\right)$

A choice correspondence F on V is said to be strictly envy-free if the above holds for all non-empty $S \subseteq Q \setminus \{i\}$.

<u>Observation 1:-</u> If F is individually rational from equal division, then F is average envy-free. This result has been noted in Thomson [1982]. However, there exists economies such that an allocation is average envy-free but is not individually rational from equal division [see Figure 1].

<u>Observation 2:-</u> Strict envy-freeness is a property due to Zhou [1992]. As observed in Thomson [1994], if we have a choice correspondence which satisfies consistency and average envy-freeness then it is also strictly envy-free.

We now define the equal income market equilibrium choice correspondence G as follows:

Let $E = \langle (u^i)_{i\in Q}\, \omega \rangle \in \mathscr{E}$. $(\overline{x}^i)_{i\in Q} \in A(E)$

is said to be an equal income market equilibrium allocation if there exists $p \in \mathbf{R}^L_+ \setminus \{0\}$ such that $\forall i \in Q$, $\overline{x}^i$ solves

$$u^i(x^i) \to \max$$

$$s.t.\ \ p.x^i \leq 1$$

$x^i \in \mathbb{R}^L_{+}$.

The following proposition is standard.

Proposition - 2:- Every $E \in \mathscr{E}$ possesses an equal income market equilibrium allocation.

In view of the above we can define G(E) to be the set of equal income market equilibrium allocations for $E \in \mathscr{E}$. G is called the equal income market equilibrium choice correspondence. Further given $E \in \mathscr{E}$, $(x^i)_{i \in Q} \in G(E)$ implies $x^i \in \mathbb{R}^L_{++}\ \forall i \in Q$.

The following proposition is easily verified:

Proposition 3:- G satisfies efficiency, Average Envy-Freeness, Strict Envy-Freeness, Consistency and Replication Invariance.

The Main Results:- We now present two variations of a theorem due to Thomson [1988].

<u>Theorem 1</u>:- Let F be a choice correspondence on $\mathscr{E}$ which satisfies Efficiency, Average Envy Freeness, Consistency and Replication Invariance. Then $\forall E \in \mathscr{E}, F(E) \subset G(E)$.

<u>Proof</u>:- Assume the conditions of the theorem for F. Let $E = \langle (u^i)_{i \in Q}, \omega \rangle \in \mathscr{E}$ and towards a contradiction assume,

$(\overline{x}^i)_{i \in Q} \in F(E) \setminus G(E)$. By proposition 1 and since, $F(E) \subset P(E)$,

there exists $p \in \mathbb{R}_+^L \setminus \{0\}$ such that $\forall i \in Q, \overline{x}^i$ solves

$$u^i(x^i) \to \max$$

$$s.t.\ \ p.x^i \leq p.\overline{x}^i$$

$$x^i \in \mathbb{R}_+^L.$$

Without loss of generality, we may assume

$p.\omega = |Q|$. We would be done if we could show $p.\overline{x}^i \le \frac{p.\omega}{|Q|}$ $\forall\, i \in Q$. Assume $p.\overline{x}^i > \frac{p.\omega}{|Q|}$ for some $i \in Q$. Thus there exists $j \in Q$ such that $p.\overline{x}^j < \frac{p.\omega}{|Q|}$. Thus $p.\overline{x}^j < p.\overline{x}^i$.

By the smoothness assumption on preferences, there exists $\overline{\lambda} \in (0,1)$ sufficiently small so that $u^j(\overline{x}^j + \lambda(\overline{x}^i - \overline{x}^j)) > u^j(\overline{x}^j)$ $\forall\, \lambda \in (0, \overline{\lambda})$. Choose $k \in \mathbb{N}$ such that $\frac{1}{1+k} < \overline{\lambda}$ and consider E^{k+1}. Let $\overline{y}^{(n,m)} = \overline{x}^n$ $\forall\, (n,m) \in Q^{k+1}$

By Replication Invariance $(\overline{y}^{(n,m)})_{(n,m) \in Q^{k+1}} \in F(E^{k+1})$. Let $S \subset Q^{k+1}$ be defined as follows:

$S' = \{(i,1)\} \cup \{(j,m) \ : \ m = 1, \ldots, k\}$.

and $S = S' \cup \{(j, k+1)\}$

Let $E' = \langle (u^n)_{n\in S'}, (k+1)\overline{x}^j + \overline{x}^i \rangle$.

By Consistency, $\left((\overline{y}^{(j,m)})_{m=1}^{k+1}, \overline{x}^i\right) \in F(E')$.

But $\dfrac{k\overline{x}^j + \overline{x}^i}{k+1} = \overline{x}^j + \dfrac{1}{k+1}(\overline{x}^i - \overline{x}^j)$. is the average consumption

of agents in S'.

Hence $u^j\left(\dfrac{k\overline{x}^j + \overline{x}^i}{k+1}\right) > u^j(\overline{x}^j) = u^j(\overline{y}^{(j,k+1)})$

This contradicts Average Envy-Freeness and proves the theorem.

Q. E. D.

As a corollary to the above theorem it easily follows that the largest choice correspondence which satisfies Efficiency, Average Envy-Freeness, Consistency and Replication Invariance is G.

In the next theorem we replace consistency and average envy freeness by strict envy freeness to obtain the following result:

Theorem 2:- Let F be a choice correspondence on $\mathscr{E}$ which satisfies Efficiency, Strict Envy Freeness and Replication Invariance. Then $\forall E \in \mathscr{E}, F(E) \subset G(E)$.

Proof:- The proof proceeds exactly as in the proof of Theorem 1, upto the construction of S'. Then we skip the construction of E' and the reference to consistency. From there on the analysis is once again the same as before, except that what now gets contradicted is strict envy-freeness (: (j, k+1) envies S').

Q.E.D.

We are thus able to characterize subcorrespondences of the equal income market equilibrium correspondence in a variable population framework, without appealing to consistency.

Reference:-

1. D. Foley [1967]: "Resource Allocation in the Public Sector," Yale Economic Essays.

2. S. Lahiri [1997a]: "Axiomatic Characterization Of Market Equilibrium Solutions In Problems Of Fair Division," mimeo.

3. S. Lahiri [1997b]: "Axiomatic Characterization Of The Equal Income Market Equilibrium Choice Correspondence In Non-Convex Economies," mimeo.

4. W. Thomson [1982]: "An Informationally Efficient Equity Criterion," Journal of Public Economics 18, 243-263.

5. W. Thomson [1988]: "A Study Of Choice Correspondences In Economies With A Variable Number Of Agents," Journal of Economic Theory, Vol.46, pages 237-254.

6. W. Thomson [1994]: "Consistent Extensions," Mathematical Social Sciences, Vol.28, pages 35-49.

7. L. Zhou [1992]: "Strictly Fair Allocations in Large Exchange Economies," Journal of Economic Theory, 57, 158-175.

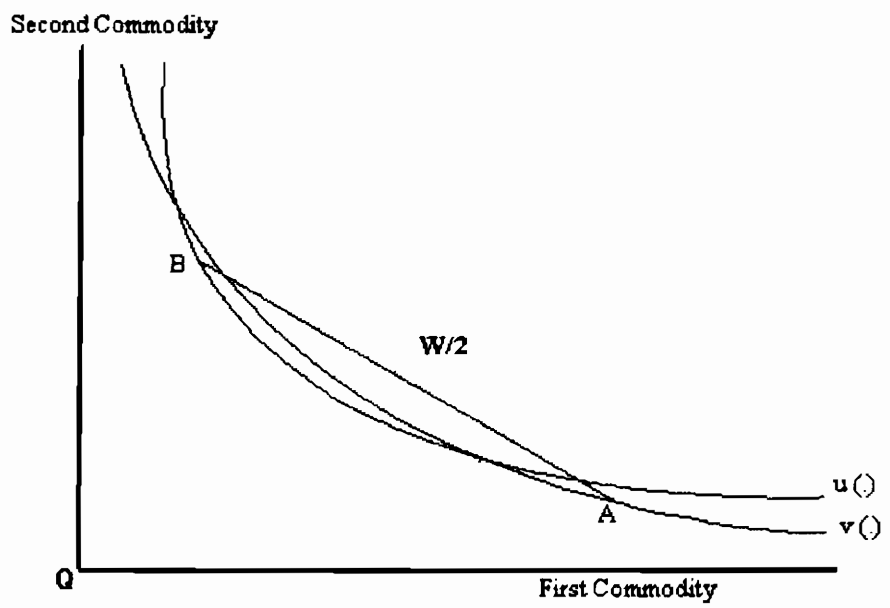


Figure 1

u (.) is the utility function of first agent.
v (.) is the utility function of second agent.

u (B) is greater than u (A). v (A) is greater than v (B). But u (W/2) is greater than u (B) and
v (W/2) is greater than v (A). Note A + B is equal to W (Social Endowment).