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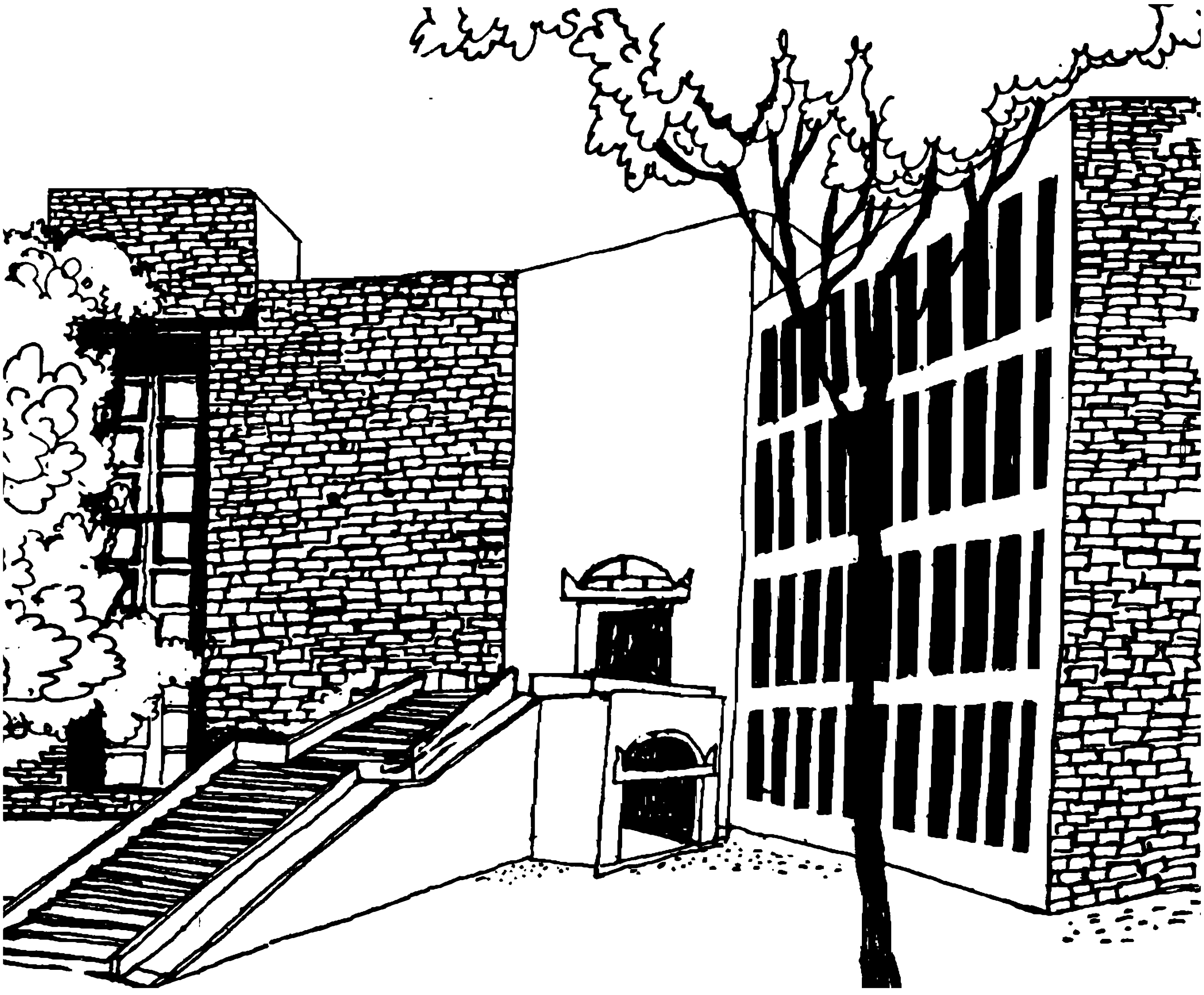

NEUTRAL TAXATION OF RISKY INVESTMENT AND PROPORTIONAL SACRIFICE

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Abstract

In this paper, our purpose is to show that a tax schedule is neutral with respect to a concave utility function if and only if it is an equal proportional sacrifice tax schedule.

1. Introduction: The problem of income taxation is an old one. A related problem is the choice of an income tax schedule, which does not lead to a diversion of scarce resources from "more to less productive employments". This is precisely the problem that Buchholz [1988] is concerned with.

In the literature on income taxation, distributive justice, in the sense of Lorenz domination is also of prime concern. There is considerable literature on this topic. Few sources are more exhaustive than the three papers of Buchholz, Richter and Schwaiger [1986], Moyes [1986] and Pfingsten [1986], all of which have been published in the same symposium volume. In the first of these three papers, we can find the concepts of equal absolute sacrifice and equal proportional sacrifice tax rules, and persuasive arguments in favour of them from the standpoint of inequality reduction.

In this paper, our purpose is to extend the result of Buchholz [1988] and show that a tax schedule is neutral with respect to a concave utility function if and only if it is an equal proportional sacrifice tax schedule.

2. The Model: We follow Buchholz [1988] in the ensuing description of the model. Investment projects with uncertain outcomes are characterized by a real valued income variable which assumes strictly positive values only. We assume that

the "states of nature in which the different outcomes of a given investment project with uncertain outcomes Z occur are well defined with known probabilities. The investor's risk preferences are given by a continuous utility function $u : \mathbb{R}_+ \rightarrow \mathbb{R}_+$ with u (0) = 0. This is a von-Neumann Morgenstern utility function for money. u is assumed to be continuously differentiable on $\mathbb{R}_{++}$ and concave with $u'(z) > 0 \ \forall z > 0$.

Risky projects are then evaluated by their expected utility. Given two investment projects with uncertain outcomes Z_1 and Z_2, the investor (weakly) prefers Z_1 to Z_2 if $Eu(Z_1) \geq Eu(Z_2)$, where E is the expectations operator.

A tax function is a function $T:\mathbb{R}_{++} \rightarrow \mathbb{R}_+$ such that $\forall y>0, T(y)<y$.

Given a Tax Function T, let $S_T(y) = y - T(y) > 0$.

We assume T is continuously differentiable with $1 \geq T'(y) \geq 0 \ \forall y > 0$. Thus $1 \geq S_T'(y) \geq 0 \ \forall y > 0$.

<u>Lemma 1</u>:- $\underset{y \to 0}{lt}\ T(y) = 0$.

Pf:- Follows easily from $0 \le T(y) < y \ \forall\ y > 0$.

<u>Q.E.D.</u>

<u>Definition 1</u>:- A tax function T is called neutral with respect to a utility function u if and only if $Eu(S_T(Z_1)) \ge Eu(S_T(Z_2))$ whenever $Eu(Z_1) \ge Eu(Z_2)$ where Z_1 and Z_2 are risky projects.

<u>Definition 2</u>: A tax function T is said to be an equal proportional sacrifice tax function with respect to the utility function u if $u(y - T(y)) = a\,u(y) \ \forall\ y > 0$ and for some $0 < a < 1$.

3. <u>The Main Theorem</u>:-

<u>Theorem 1</u>: Let u be a given utility function. A tax function T is neutral with respect to u if and only if T is an equal proportional sacrifice tax function with respect to u.

Proof: It is easy to verify that if T is an equal proportional sacrifice tax function then T is neutral with respect to u. Hence assume T is neutral with respect to u and as in Buchholz [1988] let r > 0 be chosen. Let $y_1<r<y_2$. Let $p \in (0,1)$ such that $p\, u(y_1) + (1-p)\, u(y_2) = u(r)$. Given $0<y<r$, let $f(y) > r$ be defined by,

$p\, u(y) + (1-p)\, u(f(y)) = u(r)$. By the implicit function theorem, such an f is C' and

$$f'(y) = \frac{-p}{(1-p)} \frac{u'(y)}{u'(f(y))}.$$ By neutrality,

$$pu(S_T(y)) + (1-p)\, u\big(S_T(f(y))\big) = u\big(S_T(r)\big).$$

Let $v = u \circ S_T$. Thus,

$p\, v(y) + (1-p)\, v(f(y)) = v(r)$.

Hence,

$$f'(y) = \frac{-p}{(1-p)} \frac{v'(y)}{v'(f(y))}, \quad \forall y<r.$$

Thus $$\frac{v'(y_1)}{u'(y_1)} = \frac{v'(y_2)}{u'(y_2)}.$$

Since r, y_1, and y_2 were arbitrarily chosen,

$$\frac{v'(y)}{u'(y)} = a \text{ (a constant) } \forall y > 0.$$

From this we get, $v(y) = a\, u(y) + b \quad \forall \quad y > 0$.

By continuity, $v(y) = a\, u(y) + b, \quad \forall \quad y \geq 0$.

$$\therefore u(y - T(y)) = au(y) + b \quad \forall y > 0$$

$$\therefore T(y) = y - u^{-1}[au(y) + b] \, \forall y > 0$$

Taking limits of both sides as y tends to zero, and using the facts that $u(0) = 0$, along with Lemma 1 we get $b = 0$.

Thus, $u(y - T(y)) = a\, u(y)$.

Q.E.D.

In fact full neutrality of a tax function as given in Definition 1, is somewhat more than what we require to show that the tax function is an equal proportional sacrifice tax function. It turns out that the following weaker notion of neutrality is sufficient for our purposes:

Definition 3: A tax function T is called weakly neutral with respect to a utility function u if and only if

$Eu(S_T(Z_1)) \geq Eu(S_T(Z_2))$ *whenever* $Eu(Z_1) > Eu(Z_2)$ where

Z_1 and Z_2 are risky projects.

We now require that if Z_1 is preferred to Z_2 before tax then it should not be the case that Z_2 is preferred to Z_1 after tax; indifference in the post tax situation is however permitted.

<u>Theorem 2</u>: Let u be a given utility function. A tax function T is weakly neutral with respect to u if and only if T is an equal proportional sacrifice tax function with respect to u.

<u>Proof</u>: To make the proof of Theorem 1 go through here, all we have to show is that if $0 < y_1 < r < y_2$ and $p \in (0,1)$ satisfies

$p\, u(y_1) + (1 - p)\, u(y_2) = u(r)$

then $p\, v(y_1) + (1 - p)\, v(y_2) = v(r)$.

Let $0 < \bar{p} < p$. Then

$$\bar{p}\, u(y_1) + (1 - \bar{p})\, u(y_2) > u(r).$$

By weak neutrality,

$$\bar{p}\ v\ (y_1) + (1 - \bar{p})\ v\ (y_2) \geq v\ (r).$$

Taking limits as $\bar{p}$ tends to p, we get

$$p\ v\ (y_1) + (1 - p)\ v\ (y_2) \geq v\ (r).$$

Similarly let $0 < y_1 < r < \bar{r} < y_2$.

Thus $p\ u\ (y_1) + (1 - p)\ u\ (y_2) < u\ (\bar{r})$.

By weak neutrality once again,

$$p\ v\ (y_1) + (1 - p)\ v\ (y_2) \leq v\ (\bar{r}).$$

Now taking limits as $\bar{r}$ tends to r, we get

$$p\ v\ (y_1) + (1 - p)\ v\ (y_2) \leq v\ (r).$$

Thus $p\ v\ (y_1) + (1 - p)\ v\ (y_2) = v\ (r)$.
All this only appeals to the continuity of u and v.

<u>Q. E. D.</u>

Conclusion:

We have thus been able to establish that neutrality of tax functions and equal proportional sacrifice are really one and the same thing. This establishes an identity of consequences between the desire for distributive justice and efficiency in portfolio choice.

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