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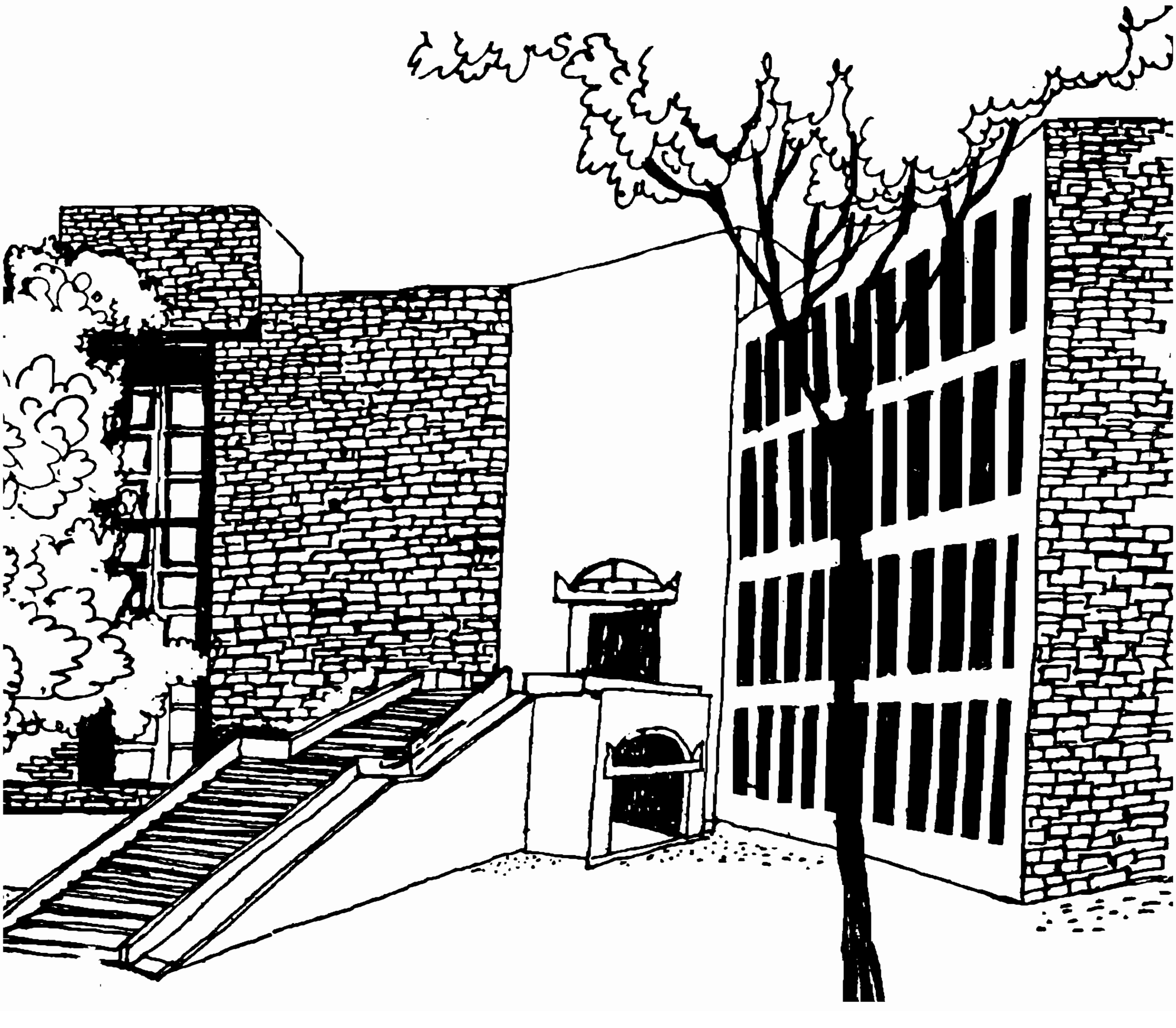

THE SUPPORTING LINE PROPERTY AND THE ADDITIVE CHOICE FUNCTION FOR TWO DIMENSIONAL CHOICE PROBLEMS

By

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Abstract

In this paper, we provide an axiomatic characterization of the additive choice function using the additivity property due to Myerson [1981]. It is seen that along with Pareto Optimality, symmetry, and a supporting line property the additivity axiom uniquely characterizes the additive choice function. This characterization appears to be a shade more elegant and less reliant on the definition of the additive choice function, than the one available in Lahiri [1997a].

It is easy to see that the additive choice function as defined in this paper, does not satisfy Nash's Independence of Irrelevant Alternatives Assumption. The latter is a property required for a choice function to be representable by an utility function i.e. the chosen point is to be the unique maximizer of an utility function. This brings us to the question of when a choice function is representable. This is the question we take up in an appendix to the paper.

1. Introduction:- A two dimensional choice problem is a compact, convex, comprehensive subset of the non-negative orthant of two dimensional Enclidean space, each such set admitting a strictly positive vector. Such a choice problem, has a natural interpretation in welfare economics as the set of feasible utility vectors for two individuals amongst whom one unit of money needs to be divided. The obvious problem confronting a decision maker, is to select a feasible utility vector and thereby arrive at an allocation of the one unit of money among the two individuals.

The most common method of arriving at such an utility allocation is by maximizing a social welfare function (i.e. a continuous real valued function defined on the non-negative orthant of two dimensional Euclidean space) subject to the point being a feasible utility vector. The literature on social welfare functions and its myriad axiomatic characterizations, have been surveyed in d'Aspremont [1985]. Of particular importance in this respect is the utilitarian social welfare function, a rather extensive and complete discussion of which can be found in d'Aspremont [1985] and references therein. The utilitarian social welfare function is the real valued function which maps to each utility vector its sum. The obvious implication of the detailed discussion of the utilitarian social welfare function is that, the chosen utility allocation should maximize the sum of the utilities from among the set of utility vectors.

The problem with a choice rule based on the utilitarian principle is that, even for genuine choice problems (i.e. ones that are not necessarily strictly convex), the solution set may consist of an uncountably infinite number of points, thus implying a complete breakdown of predictive ability. When the choice problems are strictly convex, no such problem arise and Myerson [1981] has a rather elegant characterization for the relevant choice rule on such a sub-domain using an additivity property. On a somewhat different domain, allowing for free-disposability of utility (in addition to a strictly convex feasible set of utility vectors) Moulin [1988] proposes an axiomatic characterization. However, on the kind of domain we have chosen for our analysis, the problem is rather ill-posed.

In order to circumvent this problem, in an earlier paper (Lahiri [1977a]), we proposed the additive choice function which given any choice problem selects the mid-point of the set of maximizers of the utilitarian social welfare function. We also proposed an axiomatic characterization of the additive choice function by modifying slightly the axiomatic characterization available in Myerson [1981]. In all fairness to a recent discussion on a similar issue, it should be pointed out that Klemisch - Ahlert [1996], mentions the additive choice function in the context of an axiomatic characterization (on a domain similar to that chosen by Myerson [1981]) which appears to be erroneous.

In this paper, we provide an axiomatic characterization of the additive choice function using the additivity property due to Myerson [1981]. It is seen that along with Pareto Optimality, symmetry, and a supporting line property the additivity axiom uniquely characterizes the additive choice function. This characterization appears to be a shade more elegant and less reliant on the definition of the additive choice function, than the one available in Lahiri [1997a].

It is easy to see that the additive choice function as defined in this paper, does not satisfy Nash's Independence of Irrelevant Alternatives Assumption. The latter is a property required for a choice function to be representable by an utility function i.e. the chosen point is to be the unique maximizer of an utility function. This brings us to the question of when a choice function is representable. This is the question we take up in an appendix to the paper.

2. <u>The Model and The Axioms</u>:- A two dimensional choice problem is a nonempty subset S of $\mathbb{R}^2_+$ satisfying the following properties:

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(i) S is compact and convex;

(ii) S is comprehensive i.e. $0 \leq y \leq x \in S \rightarrow y \in S$

(iii) $\exists x \in S$ *with* $x \gg 0$ *i. e. if* $x = (x_1, x_2)$ *then* $x_i > 0$ *for* $i = 1, 2$.

Let Σ^2 denote the class of all choice problems. Fundamental to axiomatic choice theory as developed by Nash [1950] and surveyed more recently in Thomson [1994] is the concept of a choice function.

Given $\phi \neq D \subset \Sigma^2$ a choice function is a function $F : D \rightarrow \mathbb{R}^2_-$ such that $F(S) \in S \; \forall \; S \in D$. Such a D as in the definition of a choice function is called a domain.

Let $F: D \rightarrow \mathbb{R}^2_-$ be a choice function. It is said to satisfy

(A) Weak Pareto Optimality (WPO) if

$\forall S \in D, F(S) \in W(S) \equiv \{ x \in S / y \gg x \rightarrow y \notin S \}$

(B) Pareto Optimality (PO) if

$\forall S \in D, F(S) \in P(S) \equiv \{ x \in S / y > x \rightarrow y \notin S \}$

(C) Symmetry (SYM) if whenever $S \in D$ satisfies

$\{ (x_2, x_1) / (x_1, x_2) \in S \} = S$, we have $F_1(S) = F_2(S)$

(D) Weak Translation Covariance (WTC) if given $S \in D$ we have

$T = \{ y \in \mathbb{R}^2_- / y \leq x + c$ *for some* $x \in S \}$ with $c \in \mathbb{R}^2_{--}$, then

F(T)=F(S)+c, whenever $T \in D$

Note:- Unlike in Translation Covariance we require here that $C \in \mathbb{R}^2_{--}$.

Additivity: If $\forall S, T \in D$ if $S+T \in D$, then F (S +T) = F (S) + F (T)

Given $S \in \Sigma^2$, let $U(S) = \{x \in \mathbb{R}^2_- \ / \ x_1+x_2 \geq y_1+y_2 \ \forall \ (y_1, y_2) \in S\}$

Let $a_1(S) = \max\{ x_1 \ / \ (x_1, x_2) \in U(S) \}$

$b_1(S) = \min\{ x_1 \ / \ (x_1, x_2) \in U(S) \}$

Let $a_2(S)$ and $b_2(S)$ be such that $a(S) = (a_1(S), a_2(S))$ and $b(S) = (b_1(S), b_2(S))$ both belong to P(S).

Define $A(S) = \frac{1}{2} [a(S) - b(S)]$.

The choice function $A: \Sigma^2 \rightarrow \mathbb{R}^2_-$ is called the additive choice function.

Before we close this section, let us note the following lemma:

Lemma 1: Let $F: D \rightarrow \mathbb{R}^2_-$ be a choice function satisfying PO and Additivity. Then F satisfies Weak Translation Covariance.

Proof: Let $S \in D, c \in \mathbb{R}^2_{--}$ and $R = \{x \in \mathbb{R}^2_- / x \leq c\}$.

Then $S+R = \{x \in \mathbb{R}^2_- \ / \ x \leq y+c$ such that $y \in S\}$.

By PO, $F(R) = c$. By Additivity, $F(S+R) = F(S) + F(R) = F(S) + c$.

<u>Q.E.D.</u>

<u>3.</u> <u>The Characterization Theorem For The Additive Choice Functions</u>:-

Let $\bar{D} = \{ S \in \Sigma^2 \ / \ a(S) = b(S) \}$ and define $\bar{A} : \bar{D} \to \mathbb{R}^2_-$ as follows:

$\bar{A}(S) = A(S) \ \forall \ S \in \bar{D}$

$\bar{A}$ is the restriction of A to $\bar{D}$.

We have the following theorem due to Myerson [1981]:

<u>Theorem 1</u>:- The only choice function on $\bar{D}$ to satisfy PO, SYM and Addi is $\bar{A}$.

In order to characterize the additive choice function $A : \Sigma^2 \to \mathbb{R}^2_-$ we need the following property in addition to the ones used in Theorem 1.

Let $F : D \to \mathbb{R}^2_-$ be a choice function. It is said to satisfy a Supporting Line Property (SLP) if $\forall \ S \in D$, there exists a $p \in \mathbb{R}^2_- \setminus \{0\}$ such that

(i) $p.x \leq p.F(S) \ \forall \ x \in S$

(ii) $T \epsilon D$ with $S \subset T$ and if $x \epsilon T \setminus S$ implies $p.x < p.F(S)$, for a p satisfying (i) then $F(T) \geq F(S)$.

This property is similar in spirit to the Independence of Irrelevant Expansions assumption due to Thomson [1981] and its variant - the Weak Irrelevant Expansions assumption due to Peters [1986] (:discussed in detail in Lahiri [1997a]). The Thomson [1981] assumption as well as its variant were used to characterize the family of non-symmetric Nash Choice functions:

Armed with this property, we can assert the following:

Theorem 2:- The only choice function on Σ^2 to satisfy PO, SYM, Addi and SLP is A.

Proof:- That A satisfies all the assumptions (and in particular SLP with p = (1,1)) is obvious. So let us prove the converse.

Let $F:\Sigma^2 \rightarrow \mathbb{R}^2$ be a choice function which satisfies the above assumptions. By theorem 1, F (S) = A (S) $\forall S \epsilon \bar{D}$. Hence assume $S \epsilon \Sigma^2 \setminus \bar{D}$ i.e. a (S) $\neq$ b (S).

By Lemma 1, (and hence WTC) we may assume $A_1(S) = A_2(S)$. Thus $a_1(S) = b_2(S)$ and $a_2(S) = b_1(S)$.

Let T = cch. { a (S), b(S) }, where cch stands for comprehensive convex hull.

By PO and SYM, F(T) = A(S). Further, the only $p \in \mathbb{R}^2_-$, which satisfies condition (i) of the SLP property is p=(1,1).

Now $T \subset S$ and $x \in S \setminus T \rightarrow x_1 - x_2 < F_1(s) + F_2(s)$.
Thus, by SLP property and PO F(S) = F(T) = A(S).

<u>Q.E.D.</u>

4. Conclusion:

In this paper we have obtained an axiomatic characterization of the additive choice function using additivity and a supporting line property. The supporting line property appears to be quite weak. Infact the family of non-symmetric Nash choice functions as well as the two benevolent dictator choice functions (:see Thomson (1994) for definitions of the benevolent dictator choice functions) all satisfy both Pareto optimality as well as the supporting line property. Hence our axiomatic characterization may be considered quite reasonable in light of current literature on the subject.

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Appendix

What implies SARP for two dimensional Choice Probelms?

Given $p \in \mathbb{R}^2_{++}$, let $S(p) = \{ x \in \mathbb{R}^2_+ / p.x \leq 1 \}$.

Let $C = \{S(p) / p \in \mathbb{R}^2_{++}\}$.

(is called the class of all (two-dimensional) linear (competitive) choice problems.

Given a choice function $F : D \to \mathbb{R}^2_+$, and $x, y \in \mathbb{R}^2_+, x \neq y$ write $x \, R_F \, y$ if and only if $\exists \, S \in D$ such that x = F (S) and y $\in$ S.

A choice function $F : D \to \mathbb{R}^2_+$, is said to satisfy the

(a) Weak Axiom of Revealed Preference (WARP) if R_F is assymetric;

(b) Strong Axiom of Revealed Preference (SARP) if R_F is acyclic.

A choice function $F : D \to \mathbb{R}^2_+$, is said to satisfy Nash's Independence of Irrelevant Alternatives Assumption (NIIA) if $\forall S, T \in D, S \subset T, F(T) \in S \Rightarrow F(S) = F(T)$.

Theorem 3 : (Rose [1958]): Let $F : C \Rightarrow \mathbb{R}^2_+$ be a choice function such that $F(S) \in P(S)\ \forall\ S \in C$. Then F satisfies WARP if and only if F satisfies SARP.

Theorem 4 (Very Easily Proved): A choice function $F : \Sigma^2 \Rightarrow \mathbb{R}^2_+$ satisfies WARP if and only if it satisfies NIIA.

For the subsequent property we refer to Thomson [1981]:

Given, $D \subset \Sigma^2$, $D \neq \phi$ and $F : D \Rightarrow \mathbb{R}^2_+$,

let $p(F,S) = \{p \in \Delta \,/\, p.x \leq p.F(S)\ \forall x \in S\}$, where

$\Delta = \{(p_1, p_2) \in \mathbb{R}^2_+ / p_1 + p_2 = 1\}$.

A choice function $F : D \Rightarrow \mathbb{R}^2_+$ is said to satisfy Independence of Irrelevant Expansions (IIE) if $\forall\ S \in D$, there exists $p \in p(F,S)$ such that whenever $S \subset T \in D, F(S) \in T$ and $p.x \leq p.F(S), F(T) = F(S)$.

This property and a weaker version of the same has been studied in Lahiri [1997a]. It is easy to see that whenever S satifies P O and I I E, and S $\in$ D, then $p \in p(F,S) \Rightarrow p >> 0$.

We do not need I I E in its full generality, but something much weaker that P O and IIE implies:

A choice function $F : D \Rightarrow \mathbb{R}^2_+$ with $C \subset D$ is said to satisfy Partial

Independence of Irrelevant Expansions (PIIE) if $\forall S \in D$, there exists S (p) ϵ C such that $S \subset S(p)$, and $F(S(p)) = F(S)$.

If F satisfies P O and PIIE, then F (S) ϵ P (S (p)) for such an S (p) ϵ C.

Theorem 5:- Let $F : \Sigma^2 \Rightarrow \mathbf{R}^2_+$ be a choice function which satisfies P O, NIIA and PIIE. Then F satisfies SARP.

Proof:- Suppose not. Then there exists $x^0, x^1, \ldots, x^n$ with $x^j \, R_F \, x^j \; \forall \, j \in \{1, \ldots, n\}$ and $x^n \, R_F \, x^0$. By NIIA and Theorem 4, n>1. Infact it is possible to prove that NIIA implies n>2. However, suppose, $n \geq 2$. Then there exists $S^0, \ldots, S^n \in \Sigma^2$, such that $x^j = F(S^j) \; \forall \, j = 0, \ldots, n, x^j \neq x^{j+1} \in S^j \; \forall \, j = 0, \ldots, n-1$ and $x^0 \in S^n$.

By PIIE. there exist $S(p^j) \in C$ such that $S^j \subset S(p^j)$ and $F(S(p^j)) = x^j \; \forall \, j = 0, \ldots, n$.

By PO, $x^j \in P(S(p^j)) \; \forall \, j = 0, \ldots, n$.

Now $x^{j+1} \in S^j \; \forall \, j = 0, \ldots, n-1$ implies $x^{j+1} \in S(p^j) \; \forall \, j = 0, \ldots, n-1$ $x^0 \in S^n$ implies $x^0 \in S(p^n)$.

But by Theorem 3, this is not possible.

Hence F satisfies SARP.

<u>Q.E.D.</u>

It is easy to see that NIIA does not imply PIIE (and hence does not imply IIE).

Infact we can now make a much stronger statement. A choice function $F : \Sigma^2 \to \mathbb{R}^2_+$ is said to be <u>representable</u>, if there exists a real valued function V on $\mathbb{R}^2_+$ such that

$\forall\, S \in \Sigma^2,\ \{F(S)\} = \{x \in S / V(x) \geq V(y)\ \forall\, y \in S\}$.

Given a choice function $F : \Sigma^2 \to \mathbb{R}^2_+$, define $f : C \Rightarrow \mathbb{R}^2_+$ by

$f(S(p) = F(S(p))$ *whenever* $p \in \mathbb{R}^2_{++}$.

Suppose F satisfies P O. Let $S = \{\, y \in \mathbb{R}^2_+ / y \leq x \,\}$, for x >> 0. Then F (S) = x. Thus $\mathbb{R}^2_{++} \subset$ range (F).

If F satisfies NIIA, then F satisfies WARP. Thus so does f. Hence by Theorem 3, f satisfies SARP.

Now by the theorem in Lahiri [1997b], there exists a function $V : \mathbb{R}^2_+ \Rightarrow \mathbb{R}$ which is uppersemicontinuous on $\mathbb{R}^2_+$, strictly monotonically increasing and strictly quasi-concave on $\mathbb{R}^2_{++}$, such that $\forall\, S \in C,\ \{F(S)\} = \{x \in S / V(x) \geq V(y)\}$. Thus we have the following theorem:

Theorem 6:- Let $F : \Sigma^2 \Rightarrow \mathbb{R}^2_+$ be a choice function which satisfies PO, NIIA and PIIE. Then F is representable by a function $V : \mathbb{R}^2_+ \Rightarrow \mathbb{R}$ which is uppersemicontinuous on $\mathbb{R}^2_+$, strictly monotonically increasing, and strictly quasi-concave on $\mathbb{R}^2_{++}$.

If a choice function $F : \Sigma^2 \Rightarrow \mathbb{R}^2_+$ satisfies SARP, then we can say that there exists a complete and transitive binary relation R such that $\forall S \in B$, $\{ F(S) \} = \{ x \in S / xRy \; \forall \; y \in S \}$. This is what Theorem 5 implies. Theorem 6 goes a step further. It says that there exists a function $V : \mathbb{R}^2_+ \Rightarrow \mathbb{R}$ such that the binary relation $\bar{R}$ *on* $\mathbb{R}^2_+$ defined by $x \bar{R} y \leftrightarrow V(x) \geq V(y)$ [: which is of necessity complete and transitive] helps us to define F i.e. $\{ F(S) \} = \{ x \in S / x \bar{R} y \; \forall \; y \in S \}$. These results should be contrasted with those in Peters and Wakker [1986].

A final point we would like to make in this appendix is that nowwhere in the proofs of Theorems 5 and 6 are we using everything that NIIA (or WARP for that matter) implies. Infact both our theorems remain intact if instead of NIIA we use a property called Restricted Weak Axiom of Revealed Preference (RWARP):

A choice function F on Σ^2 is said to satisfy RWARP if whenever $x^0 = F(S(p^0))$, $x^1 = F(S(p^1))$, $S(p^0) \in C$, $S(p^1) \in C$ and $x^0 \; R_F \; x^1$ then it is not the case that $x^1 R_p x^s$.

Infact, for a choice function F on Σ^2, P O, PIIE and RWARP trivially implies NIIA. Thus, really the only requirement beyond those in Rose [1958], to extend SARP from C to B is the PIIE property.

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