Performance of Nonparametric Control Charts

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ABSTRACT

The well known Shewhart's control chart for mean is constructed under the assumption that the distribution of underlying quality characteristic is normal. It is known that the performance of this control chart is seriously degraded if the underlying distribution is different from normal. Since in many real life situations the distribution of the quality characteristic is not known several authors have suggested use of nonparametric control charts. In this paper we use simulation to study the performance of three of these nonparametric control charts: the median chart (Janacek and Meikle, 1997), the boostrap control chart (Liu and Tang, 1996) and Hodges-Lehmann control chart (Alloway and Raghavachari, 1991) for different distributions of the quality characteristic. The in-control and out-of-control run length properties of these charts are compared against that of the control chart for mean with estimated control limits. The run length properties of all these control charts are found to be dependent on the reference sample drawn. The Hodges-Lehmann control chart is seen to perform best among all the charts considered in this study.

Keywords and Phrases: Nonparametric control chart, Median chart, Bootstrap control chart, Hodges-Lehmann control chart, ARL

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* This work was done during the second and third authors' visit to Indian Institute of Management, Ahmedabad.

1 Introduction

Control chart is one of the simplest statistical tools for monitoring quality of products particularly in the context of mass production. Despite more than seventy years of existence we continue to see an increasing number of applications of control charts not only in manufacturing, the context in which it was originally introduced, but also in other areas like finance, accounting, health administration etc. The basic aim of control charts is to detect as quickly as possible a change in the probability distribution of the quality characteristic.

The control chart was originally introduced by Shewhart under the assumption that the underlying quality characteristic is normally distributed. The mean (or $\bar{X}$)-chart is designed to detect a change in the expected value μ whereas the range (R-) chart is designed to detect a change in the standard deviation σ. The designs of these charts are such that they can be used easily on the shop floor by the employees with little training. Since its introduction the method of control charting has been extensively studied and many different kinds of charts have been proposed like, CUSUM charts (Page, 1954) for detecting small shifts in μ, EWMA charts etc. Montgomery(2000) gives an excellent introduction to the theory and applications of various types of control charts.

In recent years several nonparametric Shewhart type control charts has been proposed which do not assume any specific distribution of the quality characteristic. Some of these charts are much more complex than the Shewhart charts but with increasing use of computing technology in the shop floor it is now possible to use such charts in the shop floor without much difficulty. But before adopting any such chart it is crucial to know their properties so that one can decide which one of them is best suited for meeting the organization's goals. These charts being relatively new not much are known about their relative performance. A comparative study of performance of some bootstrap control charts have been reported in Jones and Woodall (1998). A good overview of the literature on nonparametric control charts is given in Chakraborti et al. (2001).

We consider three nonparametric control charts in this comparative study - the control chart based on the median (Janacek and Meikle, 1997), bootstrap control chart (Liu and Tang, 1996), and the control chart based on the Hodges-Lehmann estimator (Alloway and Raghavachari, 1991). We call these charts M-chart, B-chart and H-L chart respectively. We compare the performance of these charts with that of the $\bar{X}$ -chart with estimated control limits. We compare their performance based on several criteria derived from their in-control and out-of-control run length distributions obtained through extensive simulation for several symmetric unimodal distributions of the quality characteristic.

The paper is organized as follows. In section 2, we briefly discuss the construction and operation of three nonparametric control charts. In section 3, we report findings of the comparative study of the three nonparametric control charts vis-à-vis the $\bar{X}$-chart with estimated control limits. In section 4, we discuss the results obtained in section 3, and in section 5 we make some concluding remarks.

2 The Nonparametric Control Charts

In this section we briefly discuss the construction and operation of the three nonparametric control charts considered in this study. The performance of these charts are compared against that of the $\overline{X}$-chart constructed with the normal distribution as the distribution of the quality characteristic. In practical applications the control limits of the $\overline{X}$-chart is not known and needs to be estimated from the data. We estimate the control limits of the $\overline{X}$-chart on the basis of a reference sample collected when the process is in-control. The maximum likelihood estimates (MLE) of μ and σ of the normal distribution based on the reference sample are substituted in the expression of the control limits of the $\overline{X}$-chart to obtain the estimated control limits. It may be noted that due to the invariance property of the MLE these are consistent estimates of the true control limits. In what follows by $\overline{X}$-chart we mean the $\overline{X}$-chart with the control limits estimated as above.

2.1 The Median Chart (M-chart)

We discuss briefly the construction and operation of the M-chart following Janacek and Meikle (1997). Assuming the process is in in-control state we take a reference sample of size *N*. Suppose $x_i, i = 1,..., N$ are the measured values of the quality characteristic of interest for the *N* items in the reference sample. Let

$x_{(1)} \le x_{(2)} \le ... \le x_{(N)}$ be the *N* observations arranged in ascending order. For process monitoring purposes, rational subgroups of size *n* are taken at scheduled times and the median of the rational subgroups are plotted on the control chart. The upper control limit (UCL) and lower control limit (LCL) are determined using the reference sample as follows:

Let α be the probability of 'false alarm'. The LCL and UCL are $x_{(j)}$ and $x_{(N-j+1)}$ where *j* is determined from the relation

$$\frac{\alpha}{2} = \sum_{b=[\frac{n}{2}]+1}^{n} \frac{\binom{j+b-1}{b}\binom{N+n-j-b}{n-b}}{\binom{N+n}{n}} \qquad \text{...(1).}$$

Let $\hat{m}_k$ be the median of the k^th^ rational subgroup. With LCL and UCL determined as above, $P(x_{(j)} < \hat{m}_k < x_{(n-j+1)}) = 1-\alpha$, when the process is in in-control state as required. The process is suspected to be out-of-control if $\hat{m}_k$ falls outside the band determined by the LCL and the UCL.

2.2 The Bootstrap Control Chart (B-chart)

We now briefly discuss the construction and operation of the B-chart for i.i.d samples following Liu and Tang (1996). We assume that the rational subgroup size is n. For construction of the B-chart, at first a reference sample of size N is drawn when the process is in-control. Let $\overline{\overline{X}}$ be the mean of the reference sample. The following steps are repeated a large number (K) times:

(i) A random sample of size n is drawn from the reference sample using simple random sampling with replacement (SRSWR).

(ii) The mean $\overline{X_i}$ of the random sample is calculated.

(iii) Set $W_i = \sqrt{n}(\overline{X_i} - \overline{\overline{X}})$

Let $W_{(1)} \leq W_{(2)} \leq ... \leq W_{(K)}$ be the values of W_i arranged in increasing order and α be the probability of false alarm. Further let, $\tau_{\frac{\alpha}{2}} = W_{\left(\left[\frac{K\alpha}{2}\right]\right)}$ and $\tau_{1-\frac{\alpha}{2}} = W_{\left(\left[K\left(1-\frac{\alpha}{2}\right)\right]\right)}$. The LCL and UCL of the B-chart are then given by :

$$\text{LCL} = \overline{\overline{X}} + \frac{\tau_{\frac{\alpha}{2}}}{\sqrt{n}}$$

$$\text{UCL} = \overline{\overline{X}} + \frac{\tau_{1-\frac{\alpha}{2}}}{\sqrt{n}}$$

For process monitoring purposes rational subgroups are chosen at predetermined time intervals and their sample mean is computed. Let $\overline{X}_k$ be the mean of the kth rational subgroup. The process is suspected to be out-of-control if $\overline{X}_k$ falls outside the band determined by the LCL and the UCL.

2.3 Hodges-Lehmann Control Chart (H-L chart)

Alloway and Raghavachari(1991) introduced a nonparametric control chart based on the Hodges-Lehmann estimator. We briefly discuss below the construction and operation of this control chart. The Hodges-Lehmann estimator $\hat{\theta}$ for the point of symmetry θ of a continuous symmetric distribution is defined as follows:

Let $X_1, X_2, ..., X_n$ be a random sample. Define the $M = \frac{n(n+1)}{2}$ Walsh averages $W_r = \frac{X_i + X_j}{2}$, $r = 1,2,...,M$, $i \leq j, i, j \in \{1,2,...,n\}$. The Hodges-Lehmann estimator $\hat{\theta}$ is defined as the median of the Walsh averages of the sample. Suppose we collect k rational subgroups each of size n from the process when it is in in-control state. Using Table 1 of Alloway and Raghavachari(1991) we obtain the lower value L_i and the upper value U_i of the Walsh averages for each subgroup. The UCL of the H-L control chart is the median of the U_i's and the LCL is the median of the L_i's. If the Hodges-Lehmann estimator for the subgroups are $HL_1, ..., HL_k$ then the average of the HL_i's give the center line of the H-L control chart. For process monitoring purposes rational subgroups are collected at predetermined time intervals and the Hodges-Lehmann estimator is computed for every rational subgroup. A process is suspected to be out-of-control if the Hodges-Lehmann estimator of a rational subgroup falls outside the band determined by the LCL and UCL.

3 Performance of Nonparametric Control Charts

We perform a simulation based comparison of the performance of the three nonparametric control charts and the $\overline{X}$-chart. We use a reference sample of size 100 for estimation of LCL and UCL of the M-chart, B-chart and $\overline{X}$-chart. For H-L chart we use a reference sample size of 99. The false alarm rate is taken to be $\alpha = 0.0027$ which is the false alarm rate for the widely used Shewhart $\overline{X}$-chart with known control limits. The rational subgroup size is taken as 5 for the M-chart, B-chart and $\overline{X}$ -chart and 11 for the H-L chart. In this context note that H-L I chart with $\alpha = 0.0027$ can be constructed only if the rational subgroup size is at least 10.

In this comparative study we consider five symmetric unimodal distributions four of which have expectation 0 and standard deviation 1. They are normal(0,1), uniform($-\sqrt{3},\sqrt{3}$), $\frac{t_3}{\sqrt{3}}$ and double exponential (0, $\frac{1}{\sqrt{2}}$). The fifth distribution Cauchy (0,1) does not have finite expectation but has median 0 and median absolute deviation (MAD) 1. We will denote these five distributions as N, U, T, DE and C in the remaining part of this paper. We analyze the performance of each of the four control charts for each of the above five distributions of the quality characteristic.

We obtain the run length distributions for each of the four control charts when the process is in in-control state and also when it is in out-of-control state using simulation. A process can be in out-of-control state in several ways. We assume that the out-of-control state is due to a shift in the process mean. We measure the shift in process mean (δ) in terms of standard deviation (MAD in case of Cauchy) units.

We now discuss the methodology followed by us in constructing the distributions of run length when the process is in in-control and when it is out-of-control states. We discuss it with specific reference to the median control chart with Cauchy distribution as the distribution of the quality characteristic. A reference sample of size 100 is generated from C(0,1). The LCL and UCL of the M-chart is calculated based on this reference sample. Rational subgroups of size 5 are generated from C(δ,1) and their medians are calculated. The number of subgroups generated until one whose median falls outside the LCL or UCL is found, is the run length. We simulate 10000 such run lengths and based on these, (approximate) values of average run length ($ARL_1(\delta)$) and the percentiles of the distribution of run length are obtained. The methodology is similar for other control charts and distributions of the quality characteristic included in this study. Note that if $\delta = 0$ then we get the distribution of in-control run length (ARL_0).

We compare the control charts based on several criteria. They are (a) ARL_0 - the average run length when the process is in in-control state (b) $q_{0.1}^{(0)}$ - 10th percentile of the distribution of in-control run length (RL_0) (c) $ARL_1(\delta), \delta = 1,2,3$ - the average run length when the process is in out-of control state and the process median has shifted by δ (d) $q_{0.9}^{(\delta)}$, $\delta = 1,2,3$ - 90th percentile of the distribution of out-of-control run length ($RL_1(\delta)$) when process median has shifted by δ. For a control chart to be useful in practice it is desired that ARL_0 should be large, $q_{0.1}^{(0)}$ should be large, $ARL_1(\delta), \delta = 1,2,3$ all should be small and $q_{0.9}^{(\delta)}$, $\delta = 1,2,3$ all should be small.

3.1 In-control Average Run Length

The in-control average run length (ARL_0) is one of the most popular measures of performance of control charts. For the Shewhart $\overline{X}$ -chart with known control limits $ARL_0 \cong 370$, which is widely accepted as the desired ARL_0 value of control charts. Chen(1997) pointed out that the ARL_0 for $\overline{X}$ -chart with estimated control limits is different from 370 even for the normal distribution. We give the ARL_0 values of the four control charts considered in this study for the various distributions of the quality characteristic in Table 1 below. We see that though each of these control charts is nonparametric but still there is wide variation in their ARL_0 values across the various distributions. We find that ARL_0 values of the $\overline{X}$ -chart is very low for t, double exponential and Cauchy distributions and substantially lower than 370 for normal distribution. The B-chart has very low ARL_0 values for all the distributions. The M-chart has ARL_0 values greater than 370 for normal and uniform distributions and ARL_0 values lower than 370 for the other three distributions. The H-L chart has very high ARL_0 values for all distributions except Cauchy. The M-chart and the H-L chart both perform better than the $\overline{X}$ -chart in terms of higher ARL_0 . The B-chart performs better than the $\overline{X}$ -chart only for double exponential and Cauchy distributions.

Since nonparametric control charts are proposed to be used in situations where the nature of distribution is unknown it may be worthwhile to look at the minimum ARL_0 for every chart across the distributions considered. We find that in this respect the M-chart does best followed by the HL-chart.

	N	U	T	DE	C
M-chart	523.8	452.9	282.9	231.8	291.6
B-chart	65.9	254.3	50.8	116.9	14.1
HL-chart	1972.9	10733.5	442.8	923.6	156.2
$\bar{X}$ -chart	206.7	364.8	46.0	49.8	12.8

Table 1: ARL_0 values of the control charts for the five distributions

3.2 10th percentile of the distribution of in-control run length

While ARL_0 value of a control chart is widely used, increasing attention is being paid to the percentiles of the run length distribution (Chakraborti et al. (2001)). Here we consider the 10th percentile of the in-control distribution of run length($q_{0.1}^{(0)}$). In an industrial set-up stoppages due to false alarm are unwelcome as it wastes useful time and hampers productivity. We expect that such incidents should not happen frequently. Thus a control chart with high $q_{0.1}^{(0)}$ value is preferred. For the Shewhart's control chart with known control limits $q_{0.1}^{(0)}$ value can be easily calculated and is seen to be 38.97.

	N	U	T	DE	C
M-chart	53	49	31	25	2
B-chart	7	26	6	13	2
HL-chart	192	1206	45	96	17
$\bar{X}$ -chart	23	37	4	5	2

Table 2: $q_{0.1}^{(0)}$ values of the control charts for various distributions

The H-L chart has best performance among the four charts under consideration in terms $q_{0.1}^{(0)}$ value. However note that the $q_{0.1}^{(0)}$ value of the H-L chart for Cauchy distribution is quite low. Note that both the H-L chart and M-chart have better $q_{0.1}^{(0)}$ values than the $\overline{X}$ -chart with estimated control limits.

3.3 Out-of-control Average Run Length

Since control charts are operated with an aim to detect out-of-control situations as quickly as possible, it is desired that the out-of-control average run length (ARL_1) be very small. However, the out-of-control average run length is a function of the magnitude of shift (δ) in the median. In Table 3 below we provide the $ARL_1(\delta)$ values of the four control charts under consideration for three values of δ , namely, $\delta = 1, 2,$ and 3 . We note that the $ARL_1(\delta)$ values for the H-L chart are quite low indicating that the chart is capable for quickly detecting moderate to large deviations. The performance of the M-chart in this regard is inferior to that of the H-L chart. Its performance for the Cauchy distribution is extremely poor. However, we shall see in table 4, section 3.4 that the $q_{0.9}^{(\delta)}$ values for the M-chart with Cauchy distribution are not very high.

		N	U	T	DE	C
M-chart	$\delta = 1$	16.0	5.1	18.0	35.5	443.1
	$\delta = 2$	1.6	1.4	2.0	1.5	568.1
	$\delta = 3$	1.0	1.0	1.0	1.0	593.4
B-chart	$\delta = 1$	2.0	4.5	8.4	5.1	14.4
	$\delta = 2$	1.0	1.1	1.6	1.1	14.0
	$\delta = 3$	1.0	1.0	1.0	1.0	13.1
HL-chart	$\delta = 1$	2.5	1.5	1.1	1.5	30.2
	$\delta = 2$	1.0	1.0	1.0	1.0	3.3
	$\delta = 3$	1.0	1.0	1.0	1.0	1.1
$\bar{X}$ -chart	$\delta = 1$	3.5	3.3	2.9	2.2	12.6
	$\delta = 2$	1.0	1.0	1.0	1.0	8.0
	$\delta = 3$	1.0	1.0	1.0	1.0	10.9

Table 3: $ARL_1(\delta)$ **values for different values of** δ

3.4 90th percentile of the distribution of out-of-control run length

The $ARL_1(\delta)$ values tell us how long we have to wait on an average for the control chart to produce an out-of-control signal when the process median has shifted by δ. However, the process engineer may actually want to know the "maximum" time that one has to wait for an out-of-control signal to be generated. While it is not possible to answer this question in an absolute sense one may attempt to answer it in a probabilistic sense. Since the probability that an out-of-control run length with shift in median δ exceeds $q_{0.9}^{(\delta)}$ with probability only 0.1 we can possibly think of $q_{0.9}^{(\delta)}$ as an "operational maximum" value of the out-of-control run length. Table 4 below gives the $q_{0.9}^{(\delta)}$ values for the four control charts for each of the five distributions. From the table we see that the H-L chart performs reasonably well for all the five distributions under consideration. The $q_{0.9}^{(\delta)}$ values of M-chart are reasonable for δ = 2, 3 for all the five distributions.

		N	U	T	DE	C
M-chart	$\delta = 1$	216	45	621	306	28
	$\delta = 2$	27	10	14	47	16
	$\delta = 3$	1	1	1	1	8
B-chart	$\delta = 1$	33	56	82	59	33
	$\delta = 2$	6	8	33	9	32
	$\delta = 3$	2	2	11	3	31
HL-chart	$\delta = 1$	5	17	1	3	70
	$\delta = 2$	1	1	1	1	7
	$\delta = 3$	1	1	1	1	2
$\overline{X}$ -chart	$\delta = 1$	51	49	41	28	28
	$\delta = 2$	8	7	14	5	27
	$\delta = 3$	8	2	5	2	28

Table 4: $q_{0.9}^{(\delta)}$ values for different values of δ

4 Discussion of the results

It appears that among the three nonparametric control charts considered in this study the H-L chart is the best. However, this may be due to the higher size of the rational subgroup used. The performance of the M-chart comes out to be next best in this study. Both these charts perform better than the $\overline{X}$ -chart. However the performance of the B-chart is not good and in some cases its performance is worse than that of the $\overline{X}$ -chart. However, during the course of the study we found that the distributions of in-control and out-of-control run lengths are severely dependent on the reference sample. The variability in ARL_0 , $ARL_1(\delta)$, $q_{0.1}^{(0)}$ and $q_{0.9}^{(\delta)}$ on account of reference sample is substantial even with a reference sample of size 100 which is quite large in practical terms. Since the reference sample may not have its mean close to the population mean the function $ARL_1(\delta)$ may not be a symmetric function about 0 as is usually the case. Thus the figures reported in section 3 above are more of an indicative nature. The computation of true ARL_0 and $ARL_1(\delta)$ of $\overline{X}$ -chart with estimated control limits for the case of normal distribution has been reported in Chen (1997). Since the method of estimating σ for our $\overline{X}$ -chart is different from the three methods discussed in Chen's paper the results obtained by him are not exactly applicable for our case. However, we can learn substantially regarding the behaviour of such charts from the results obtained by Chen. By inspecting Table 1 of Chen(1997) we find that the standard deviation of the run lengths of the $\overline{X}$ -chart with estimated control limits when the process is in-control is quite large. This indicates that the false alarm rate of this chart will be difficult to predict in any practical implementation which is a serious drawback. Chen(1998) derive and evaluate the run length distributions associated with the R, s and s^2 charts when the process standard deviation σ is estimated. To the best of our knowledge no such study has been reported in the literature for the nonparametric control charts included in this study. Such studies are extremely important for us to have a complete knowledge of the properties of the control chart prior to their implementation in actual industrial set-ups. To illustrate the serious nature of the problem we computed the UCL and the LCL of the M-chart for 20 different reference samples from the Cauchy(0,1) distribution. The results are given in the Figure 1 below.

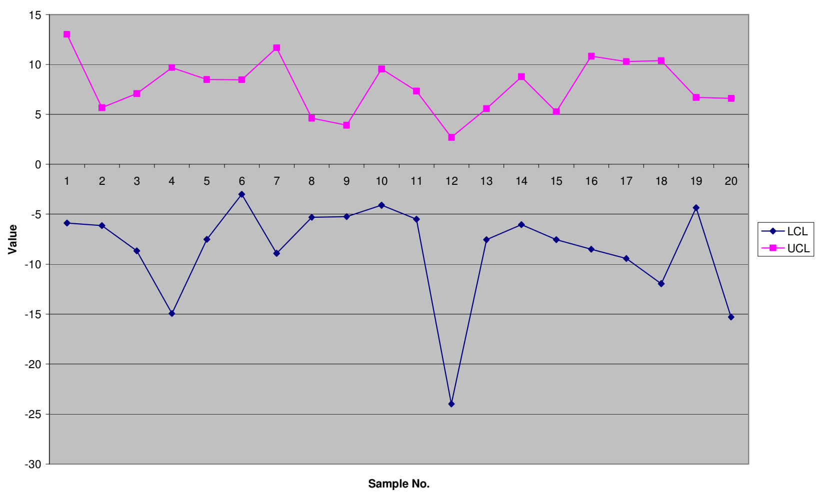

Figure 1: LCL and UCL of M-chart with Cauchy(0,1) distribution

We see from Figure 1 that the width of the in-control regions (i.e. UCL – LCL) varies from 9.1 to 26.7 for these 20 reference samples. Clearly this variation will induce large variation in in-control and out-of-control distributions of run length. Also the mid-point of the in-control region varies from -10.6 to 3.6 thereby inducing asymmetry in the values of $ARL_1(\delta)$. Thus the properties of the M-chart will depend on the reference sample in a crucial way and may not be suitable for applications which demand precise in-control and out-of-control ARL values. We carried out similar exercise for all the control charts considered in this study with the quality characteristic having normal and Cauchy distributions. We found that the control limits are more variable for Cauchy distribution than normal distribution for all the control charts. The LCL and UCL of the H-L control charts were found to be the least variable for both normal and Cauchy distributions (see Figure 2 below).

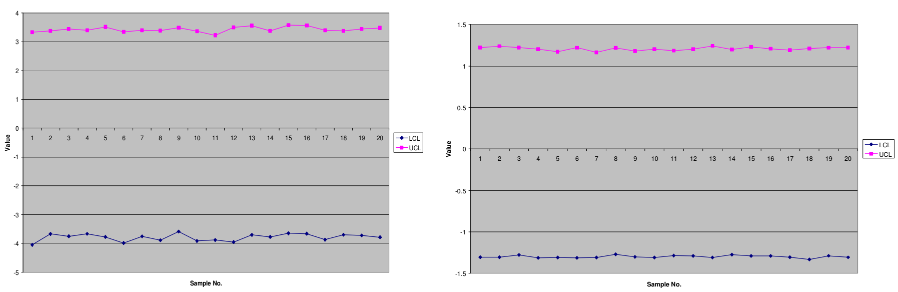

Figure 2: LCL and UCL of H-L chart with Cauchy(0,1) distribution (left) and Normal(0,1) distribution (right).

5 Concluding Remarks

The H-L chart comes out to be the most desirable of the four control charts included in this study. The LCL and UCL of this chart are less dependent on the reference sample than the other charts. Also the chart seems to have quite reasonable in-control and out-of-control run length properties as indicated by the findings of section 3. This is partially offset by the higher requirement of rational subgroup size (≥ 10) which may be difficult in practical applications. The LCL and UCL of the other charts depend on the reference sample to a much greater extent than the H-L chart. Due to this the run length properties of these charts are difficult to obtain making them unsuitable for use in practical applications. In view of all of the above we recommend use of the H-L control chart when the distribution of the quality characteristic is not known.

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