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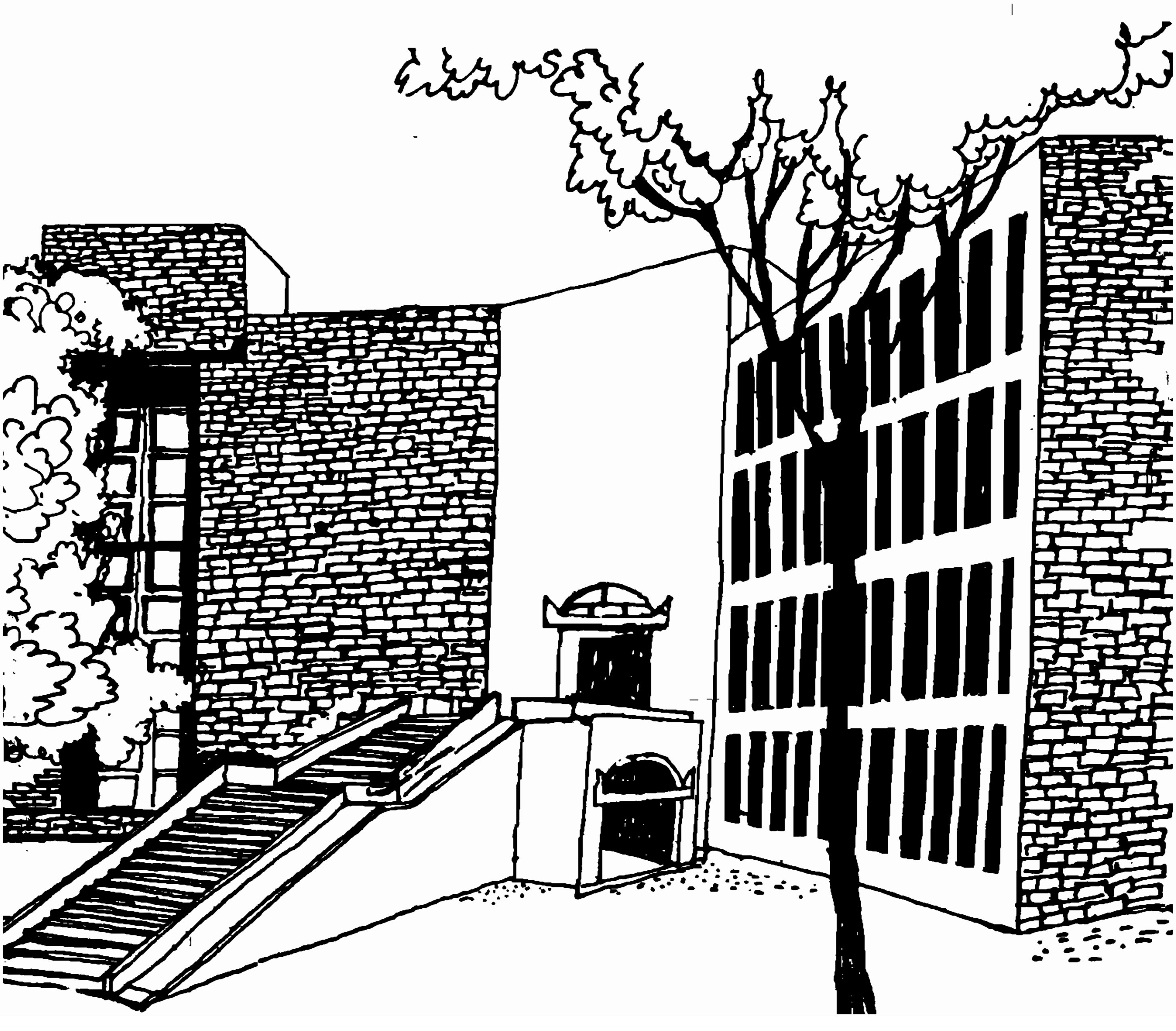

A COMMENT ON THE ADJUSTED PROPORTIONAL SOLUTION FOR RATIONING PROBLEMS

By

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Abstract

In this paper we obtain a simple expression for the adjusted proportional solution for rationing problems, when a fixed supply of an infinitely divisible good has to be divided only among two people, and when their demands exceed supply.

1. Introduction:- The bankruptcy problem, which deals with the problem of how to divide an estate among a group of creditors when the estate is insufficient to meet the debts of a deceased has an interesting origin. Jewish scholars have addressed this problem at least since the time of the collation of the Talmud: a 2,000 year old document that forms the basis for Jewish civil, criminal and religious law. Its modern day avatar is the economic theoretic literature on rationing: given a downward sloping market demand curve for a good, if price is below the equilibrium price (i.e. the price at which quantity demanded is equal to a fixed supply), then there is excess demand, and consequently a need for rationing. The problem now reduces to dividing the fixed supply among the consumers by applying some acceptable rationing rule.

O'Neill (1982) was the first person to study the bankruptcy problem using game-theoretical methods. It was followed by Aumann and Maschler (1985), who applied the nucleolus to the transferable utility game generated by a bankruptcy problem. Other contributions in the same tradition are those of Curiel, Maschler and Tijs (1988), Dagan and Volij (1993), Lahiri (1996a,b).

Of particular interest to us, in this paper is a solution called the adjusted proportional solution due to Curiel, Maschler and Tijs (1988). A slight modification of it is obtained by joining the point of minimal expectations to the point consisting of effective demands, by a straight line. The point where this straight line intersects the set of feasible allocations is the modified adjusted proportional solution. In an appendix to this paper we discuss the simple geometry determining the modified adjusted proportional solution, and we show that this solution is at one and the same time equal to the point of equal loss from effective demands as also the point of least distance from effective demands, in the two agent case.

Our position in this paper is that at least for two agent problems, the adjusted proportional solution due to Curiel, Maschler and Tijs [1988] has a much easier interpretation as the modified adjusted proportional solution.

2. The Model:- Consider a set of agents indexed by i=1, 2,...,n where n is a natural number greater than or equal to two. Let N = {1, 2, ..., n} denote the set of agents. A rationing (bankruptcy) problem is an ordered pair $(d, S) \in \mathbb{R}^n_+ \times \mathbb{R}_+$ such that $S < \sum_{i=1}^{n} d_i$.

Let B^N denote the set of all rationing problems(for N).

An allocation for $(d, S) \in B^N$ is a vector $x \in \mathbb{R}_+^n$ such

$x_i \le d_i \forall i \in N$ and $\sum_{i \in N} x_i = S$.

A solution is a function $F : B^N \to \mathbb{R}_+^n$ such that F(d,S) in an allocation for (d,S) whenever $(d, S) \in B^N$.

Given $(d, S) \in B^N$, the effective demand vector (for (d,S)) denoted d^s is the vector whose i^{th} component $d_i^s = \min \{ d_i, S \}$

Obviously, since S is what all there is for distribution any claim greater that S is as good as demanding the entire supply. Hence our definition of effective demand.

Given $(d, S) \in B^N$, the point of minimal expectation $v^{(d, S)}$ (denoted merely by v whenever there is no scope for confusion) is the vector whose i^{th} coordinate v_i is equal to $\max \left\{ 0, S - \sum_{j \ne i} d_j \right\}$ i.e. what every one else willingly concedes to i.

Observation 1: $v_i \le d_i \; \forall \; i \in$ N

Proof of observation: Suppose $v_i > d_i$ for some $i \in$N

Clearly $d_i > 0 \to v_i = S - \sum_{j \ne i} d_j$

$\therefore S - \sum_{j \ne i} d_j > d_i$

$\rightarrow S > \sum_{j=1}^{n} d_j$ which is a contradiction. Hence the observation.

Q.E.D.

Observation 2:- Given $(d, S) \varepsilon B^N$ if x is any allocation for (d,S), then $x_i \geq v_i \; \forall \; i \; \varepsilon$ N.

Proof of observation: Suppose $0 \leq x_i < v_i$ for some $i \varepsilon$N.

Then clearly $v_i = S - \sum_{j \neq i} d_j$.

$\therefore x_i < S - \sum_{j \neq i} d_j$.

$\therefore x_i + \sum_{j \neq i} d_j < S$

But $x_j \leq d_j \; \forall \; j \; \varepsilon$ N

$\therefore S = \sum_j x_j \leq x_i + \sum_{j \neq i} d_j < S$ which is a contradiction.

This proves the observation.

Q.E.D.

Observations 3:

For all $(d, S) \; \varepsilon \; B^N, \; \forall \; i \; \varepsilon$ N

$$v_i = \max \left\{ 0, S - \sum_{j \neq i} d_j^s \right\}.$$

Proof:- Let $i \; \varepsilon$ N, $k \neq i$, $k \; \varepsilon$ N.

If $d_k > S$ then $S - \sum_{j \neq i} d_j < S - d_k < 0$.

$\therefore\ v_i = 0.$

Since $d_k^s = S$ and $S - \sum_{j\neq i} d_j^s < S - d_k^s = S - S = 0$, $\max\left\{0, S - \sum_{j\neq i} d_j^s\right\} = 0.$

$\therefore\ v_i = \max\left\{0, S - \sum_{j\neq i} d_j^s\right\}.$

On the other hand if $d_k \leq S\ \forall\ k \in N,\ k \neq i,$ then

$d_k^s = d_k\ \forall\ k \in N,\ k \neq i,$ so that

$$\sum_{j\neq i} d_j^s = \sum_{j\neq i} d_j$$

This proves the observation in either case.

Q.E.D.

Observation 4:- Given $(d, S) \in B^N$, $\sum_{i\in N} v_i \leq S$

Proof of observation:- Let $x \in \mathbb{R}_+^n$ with $x_i = \dfrac{d_i}{\sum_{j\in N} d_j} S.$

It is easy to check that x is an allocation for (d,S). Thus the set of allocations for (d,S) is nonempty. Since $v_i \leq x_i\ \forall\ i \in N$ by observation 3, we have, $\sum_{i\in N} v_i \leq S.$

Q.E.D.

We now define the adjusted proportional solution $AP : B^N \rightarrow \mathbb{R}_+^n$: Given $(d, S) \in B^N$, Let

$$d_i^* = \min\left\{d_i - v_i, S - \sum_{j\in N} v_j\right\}$$

Then denoting $AP(d, S) = \overline{x}$, we get, $\overline{x}_i = v_i + \frac{d_i^*}{\sum_{j \in N} d_j^*}\left(S - \sum_{j \in N} v_j\right), \; i \in N.$

In this paper we are concerned with a modified (version of the) adjusted proportional solution, $MAP : B^N \rightarrow \mathbf{R}_+^N$, defined thus. Let $MAP(d, S) = \hat{x}$. Then,

$$\hat{x}_i = v_i + \frac{d_i^* - v_i}{\sum_{j \in N} (d_j^* - v_j)}\left(S - \sum_{j \in N} v_j\right) \forall \; i \in N.$$

This is precisely the solution that we discussed in the introduction. Unlike the adjusted proportional solution, the modified adjusted proportional solution satisfies Independence of Irrelevant Claims, a property which says that $F(d, S) = F(d^*, S) \; \forall \; (d, S) \in B^N$.

3. <u>The Two Agent Situation</u>:- We are particularly interested (in this paper) in the modified adjusted proportional solution for two agent problems i.e. for the case n=2. Without loss of generality, and for greater ease of exposition, let us assume $d_1 \leq d_2$ *whenever* $(d, S) \in B^{\{1,2\}}$ What does the adjusted proportional solution look like in this situation.

<u>Case 1</u>:- $d_1 \leq d_2 \leq S$.

Thus $v_1 = S - d_2, \; v_2 = S - d_1$.

$$d_1 - v_1 = (d_1 + d_2) - S$$

$$d_2 - v_2 = (d_1 + d_2) - S$$

$$S - v_1 - v_2 = (d_1 + d_2) - S.$$

$$\therefore \frac{d_1^*}{d_1^* + d_2^*} = \frac{1}{2}, \; i = 1, 2.$$

On the other hand,

$$d_i^s - v_i = (d_1 + d_2) - S = d_i^*, \; i = 1, 2.$$

$$\therefore \frac{d_i^s - v_i}{\sum_j (d_j^s - v_j)} = \frac{1}{2}$$

$$\therefore AP(d, S) = MAP(d, S).$$

<u>Case 2</u>:- $d_1 < S \leq d_2$

Thus $v_1 = 0, \; v_2 = S - d_1$

$$d_1 - v_1 = d_1, \; d_2 - v_2 = (d_1 + d_2) - S$$

$$S - v_1 - v_2 = d_1.$$

$$\therefore d_1^* = d_1, d_2^* = d_1 = S - v_1 - v_2, \text{ since}$$

$$d_2 - v_2 = (d_1 + d_2) - S \geq d_1 \text{ given that } d_2 \geq S.$$

$$\therefore d_1^* = d_2^* = d_1$$

$$\therefore \frac{d_i^*}{d_1^* + d_2^*} = \frac{1}{2}$$

On the other hand,

$$d_1^s - v_1 = d_1, d_2^s - v_2 = S - S + d_1 = d_1$$

$$\therefore \frac{d_i^s - v_i}{\sum_{j=1}^{2} \left(d_j^s - v_j \right)} = \frac{1}{2}$$

$\therefore AP(d, S) = MAP(d, S)$.

Case 3:- $S \le d_1 \le d_2$

$\therefore v_1 = v_2 = 0$.

$\therefore d_1 - v_1 = d_1,\ d_2 - v_2 = d_2$

$d_i^* = \min \{ d_i - v_1,\ S - v_1 - v_2 \} = \min \{ d_i, S \} = S,\ i = 1, 2$

$$\therefore \frac{d_i^*}{d_1^* + d_2^*} = \frac{1}{2}$$

$d_i^s = S,\ i = 1, 2.$

$d_i^s - v_i = S.$

$$\therefore \frac{d_i^s - v_i}{(d_1^s - v_1) + (d_2^s - v_2)} = \frac{1}{2},\ i = 1, 2.$$

$\therefore MAP\ (d, S) = AP(d, S)\ \forall\ (d, S) \in B^{(1,2)}$

We have thus essentially proved the following theorem.

Theorem 1:- For n=2, MAP = AP. Hence AP satisfies Independence of Irrelevant Claims (since MAP does so always).

We shall now explore the relation between v and d^S whenever $(d, S) \in B^{\{1,2\}}$. We have, $v_i = \max\{0, S - d_j^s\},\ j \neq i.$

$$= S - d_j^s.$$

Further, $0 \leq v_i \leq d_i$.

Thus the vector where i gets v_i and j gets d_j^s is an allocation for (d,S). Given our earlier result that $v_i \leq x_i$ whenever x is an allocation for (d,S), we get now that $v_i = \min\{x_i \,/\, x \text{ is an allocation for } (d,S)\}$ Further, $d_i^s = \max\{x_i \,/\, x \text{ is an allocation for } (d,S)\}$. Thus d^s is the north - eastern extremity and v is the southwestern extremity of the rectangle, whose diagonal (which separates d^s and v) is precisely the set of all allocations for (d,S).

[Insert Figure-1 here].

Clearly, MAP (d,S) is the mid-point of the set of all allocations for (d,S). The two extreme points of the set of all allocations for (d,S) are (v_1, d_2^s) and (d_1^s, v_2). Thus we have the following theorem:

Theorem 2:- For n =2,

$$AP(d,S) = MAP(d,S) = \left(\frac{v_1 + d_1^s}{2}, \frac{v_2 + d_2^s}{2}\right)$$

The proof of this theorem follows essentially from Theorem 1 and the observation immediately before Theorem 2.

4. Conclusion:- We have thus been able to obtain a much simpler expression for the adjusted proportional solution for two agent problems as described in Theorem 2. This simpler expression, would hopefully make the issues easier to comprehend for policy makers who may want to apply the adjusted proportional rule for rationing problems.

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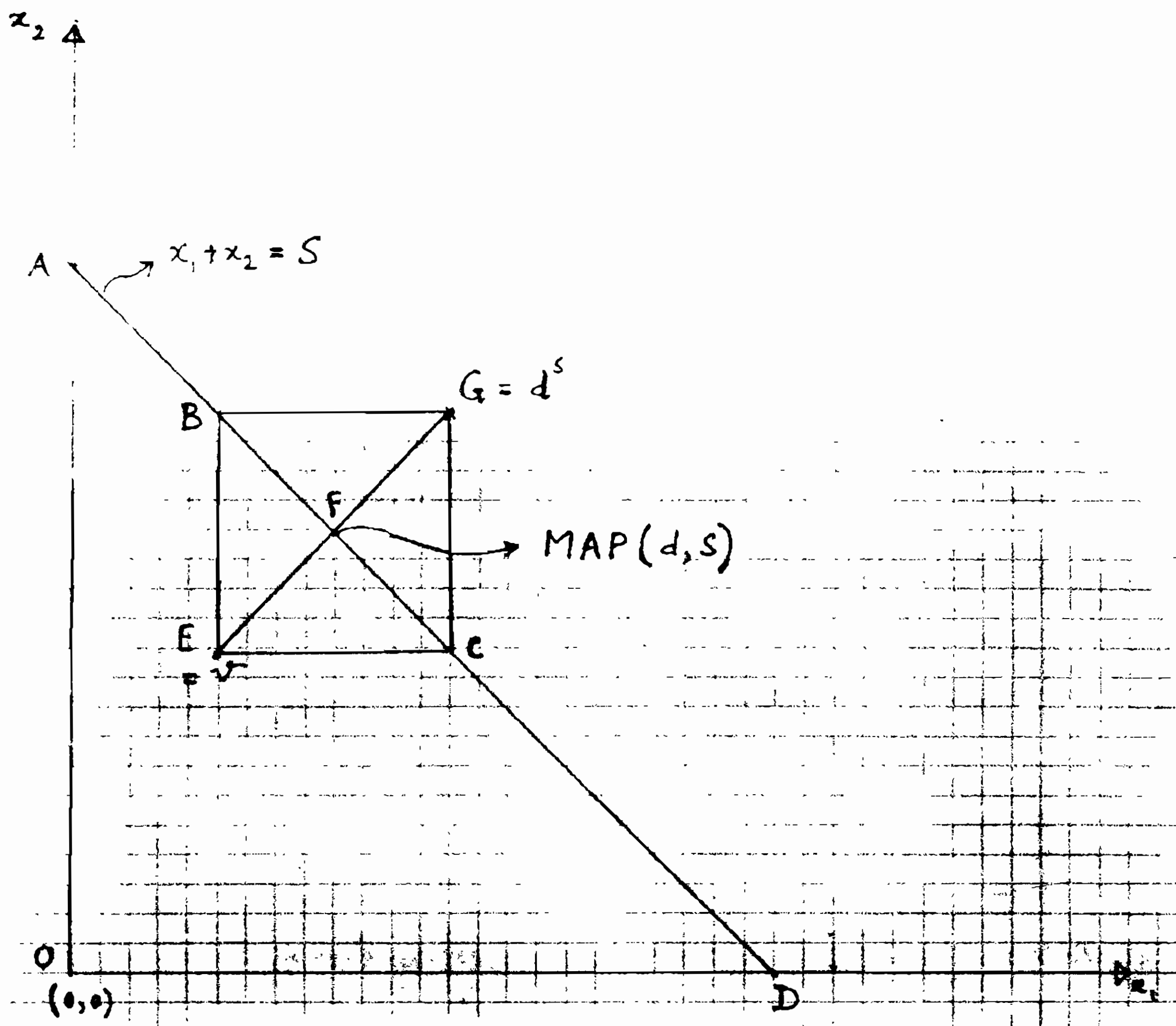

Figure 1

Appendix

The Simple Geometry of the
Modified Adjusted Proportional Solution

Let us revert to Figure 1 and show among other things that the rectangle B G C E must indeed be a square. This will lead to several equivalences.

First note that both angles ADO and angles DAO must be 45 degrees. Thus angles EBC and ECB are 45 degrees. Hence triangle EBC is an isosceles triangle. Thus the length of the side BE is equal to the length of the side EC. Thus the rectangle BGCE is indeed a square.

Hence GF must be perpendicular to the line AD. Hence F must be the point of least distance from G.

Further, it is not difficult to see that the angles FGC and BGF are both 45 degrees. Hence F is also the point of equal loss from G. Noting that F stands for MAP(d,S), we have the following theorem:

Theorem A: For n=2, given $(d, S) \in B^{\{1,2\}}$, $MAP\ (d, S)$ is the unique allocation which is at the point of least distance from d^s. Further, MAP (d,S) is also the unique allocation which equates losses among the two agents.

Given the above discussion it is easy to see that the co-ordinates of B are given by $(S - d_2^s, d_2^s)$ and the co-ordinates of C are given by $(d_1^s, S - d_1^s)$. Hence (and also from earlier discussions), $v = (S - d_2^s , S - d_1^s)$. By applying Theorem 2 in the paper we have the following result:

Theorem B:- For n = 2,

$$AP(d, S) = MAP(d, S) = \left(\frac{S + d_1^s - d_2^s}{2}, \frac{S + d_2^s - d_1^s}{2} \right).$$

This is precisely the Constested Garment Solution for the two agent case, discussed in Aumann and Maschler (1985).