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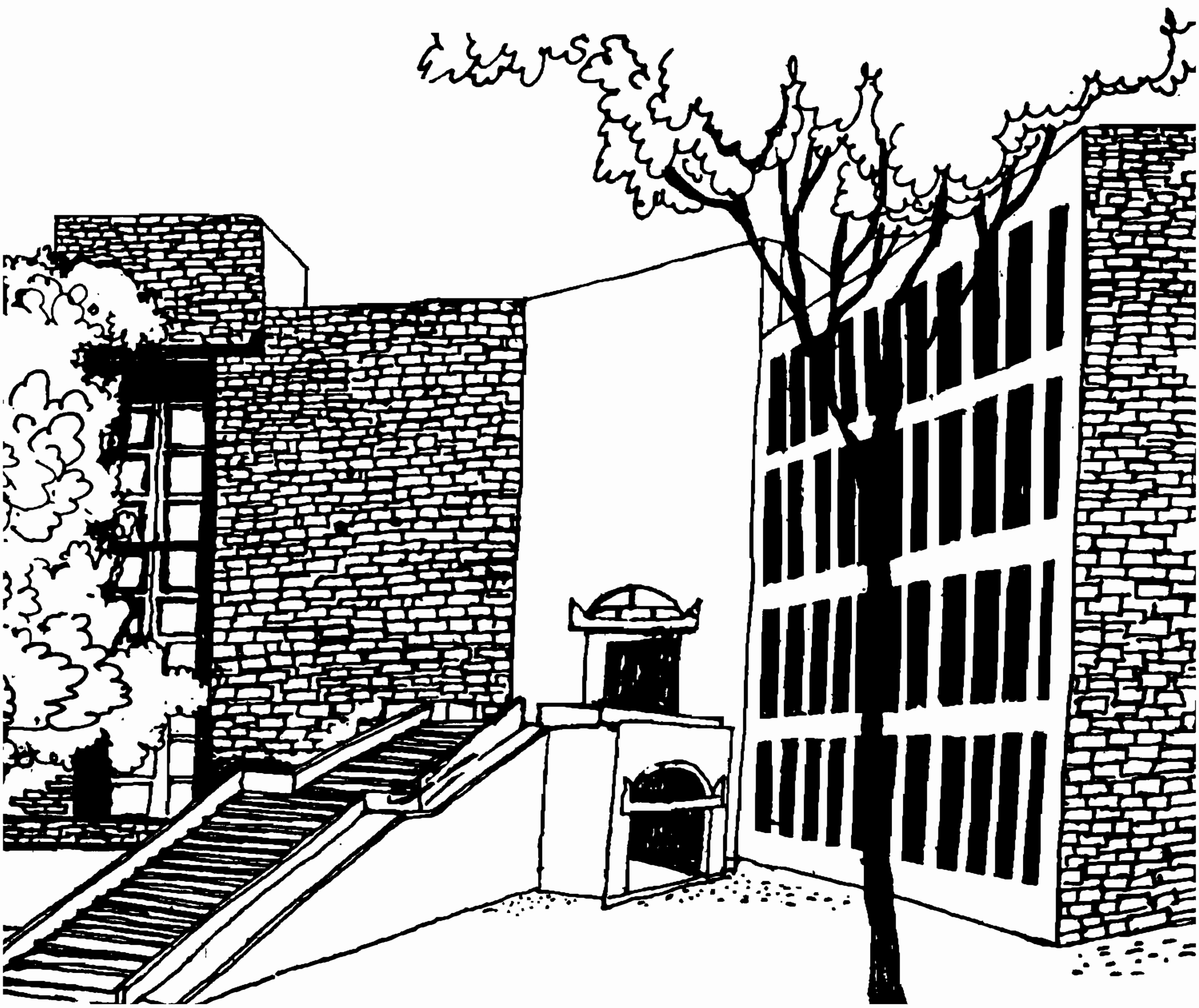

AN ALGORITHM FOR THE MIN-MAX LOSS RULE FOR CLAIMS PROBLEMS

By

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Abstract

In this paper we provide an algorithm which gives us the unique solution to the problem of minimizing the maximum loss (where loss is measured by unsatisfied demands) for a claims problem. The answer lies crucially on the structure of the problem.

Introduction:

Allocating a single homogeneous divisible good amongst a finite number of agents, when their aggregate demand exceeds the total available supply, is a problem which has precedents both in academics as well as real world situations. The literature on the subject, normally traces its origin to the Babylonian Talmud, where the problem has been characterized as a bankruptcy problem: the total estate is less than what has been willed away by a deceased ancestor to his several heirs. With this interpretation in mind, O'Neill [1982] and Aumann and Maschler [1985], provided rigorous analytical treatments of some solutions to such problems, notably the proportional solution and the contested garment solution.

The mathematical structure of the above problem resembles one of fair and just incidence of tax on a group of agents who share a common facility. Such has been the interpretation behind the literature on cost-sharing, with contributions coming from Young [1987a, 1987b, 1988], culminating in the analysis in Young [1993] and from Moulin [1985], leading to the analysis in Moulin [1988]. However, the meaning of a cost sharing problem is just the opposite of a bankruptcy problem. Whereas in a cost sharing problem, the relevant criterion is net benefit, the criterion in a bankruptcy problem is net loss. In some senses, we prefer a larger net benefit to a smaller one and a smaller net loss to a larger one; and this leads to a very different approach towards the problem as we shall shortly see in this paper.

In an earlier paper Lahiri [1996], we proposed the interpretation of a supply-chain management framework for the claims problem. The problem there was to characterise the proportional solution using a reduced game property and the paper was written in a spirit similar to that which lead to Dagan and Volij [1993], where once again concepts from bargaining game theory played a significant role. The contribution by Dagan (1996) is yet another addition to the literature on axiomatic characterizations of solutions to claims problems. In, Lahiri [1996] we offered the following interpretation: assume there is a distributor of a commodity who supplies to a finite number of retailers. Suppose the aggregate demand of the retailers is equal to the total amount available with the distributor. Then the problem is resolved very easily. Give each retailer what he desires. However, if the aggregate demand exceeds the available supply, the distributor has to ration the retailers. How does he do it?

There is one problem with this interpretation if we view our solutions as serious prescriptions in real world situations. If the retailers know that there is a shortage (as it happens when shortages are chronic) then they have strong incentives to overstate there demands for most allocation rules. The above scenario will work, only if shortages are not chronic, so that by overstating their demands, retailers would run the risk of running up unwanted inventory costs. In other words, retailers should be unaware of what the available supply is in any period, perior to the actual allocation of resources. With this caveat, the mechanism is indeed workable. With this interpretation, we are also able to retrieve the original net loss concept (associated with bankruptcy problems) through the unsatisfied demands of the retailers.

If individual benefit is the cardinal principle guiding the choice of a solution, a natural idea is to maximize the minimum benefit. On the otherhand if individual loss is the issue, then a natural way to resolve conflict for a group is to minimize maximum loss. Stated thus, the mathematical problem seems to be one of just treating one as the negative of the other and treating both problems as solved when one is. That is precisely the approach of decision theory. However, due to the peculiar structure of our problem, the above approach is simply meaningless. In fact, the two approaches are not related except superficially. (Contrast our analysis with that in Moulin [1988] where he pursues the cost-sharing problem).

In this paper we provide an algorithm which gives us the unique solution to the problem of minimizing the maximum loss for a claims problem. The answer relies crucially on the structure of the problem as the ensuing analysis reveals.

2. <u>The Model</u>: Consider a society consisting of n agents where n is a natural number. The claim of agent i (on a single infinitely divisible homogeneous good) is given by a real number $c_i>0$. The total available amount of the good is M > 0. We assume $\sum_{i=1}^{n} c_i > M$, so that the claims are incompatible.

A claims problem is an ordered pair $(c, M) \in \mathbb{R}^n_{++} x \mathbb{R}_{++}$ such that c is a vector of claims and M is the total available amount of the good. The set of all claims problems is denoted by B.

Given $(c, M) \in B$, an allocation for (c,M) is a vector $x \in \mathbb{R}^n_+$ *such that* $x_i \leq c_i \forall i$ *and* $\sum_{i=1}^{n} x_i = M$.

A solution on B is a function $F: B \rightarrow \mathbb{R}^n_+$ such that F(c,M) is an allocation for (c,M) whenever $(c,M) \varepsilon B$.

Without loss of generality and throughout the paper, we will assume that whenever we are given a claims problem $(c, M) \in \mathbb{R}^n_{++} x \mathbb{R}_{++}$ (*with* $\sum_{i=1}^{n} C_i > M$), *we have* $c_1 \leq c_2 \leq \ldots\ldots \leq c_n$. This does not affect the ensuing analysis in any way; on the contrary it simplifies matters to a great extent.

Given (c,M) ε B let k(c,M) ε {1,...,n} be defined as follows:

$$k(c,M) = 1 \ \text{if} \ c_1 \geq \frac{1}{n}\left(\sum_{i=1}^{n} c_i - M\right)$$

$$= \min\left\{k / c_{k-1} < \frac{1}{n-k+1}\left(\sum_{i=k}^{n} c_i - M\right) \leq c_k\right\}$$

Such a k(c,M) always exists.

We define the <u>quasi-equal loss solution</u> $Q: B \to \mathbf{R}^n_+$ *as follows*:

$$Q_i(c,M) = 0 \ \text{if} \ i < k(c,M)$$

$$= c_i - \frac{1}{n-k(c,M)+1}\left(\sum_{j=k(c,M)}^{n} c_j - M\right) \forall i \geq k(c,M)$$

Basically, the quasi-equal loss solution operates by allocating nothing to those whose demands are very small and then allocating the total amount among the rest in such a way that the loss experienced by each agent in the latter group is equal. Indeed, individual loss in the latter group is $\frac{1}{n-k(c,M)+1}\left(\sum_{j=k(c,M)}^{n} c_j - M\right)$. A point to be noted is that if k(c,M) = 1 (i.e. the set of agents whose index comes before k(c,M) is empty) then we have the equal loss rule.

The above rule is an algorithm and as we shall see shortly, this algorithm is the unique solution of a well defined programming problem.

3. The Main Result:-

Theorem 1: Given $(c,M)\varepsilon B$, the unique solution to the programming problem

$$\min_{x_1,\ldots,x_n}\left\{\max_{i=1,\ldots,n}\{c_i - x_i\}\right\}\ldots\ldots\ldots\ldots\ldots(1)$$

$$\text{s.t. } 0 \le x_i \le c_i \quad \forall i\ldots\ldots\ldots\ldots\ldots\ldots(2)$$

$$\sum_{i=1}^{n} x_i = M\ldots\ldots\ldots\ldots\ldots\ldots\ldots(3)$$

is Q (c,M)

The proof proceeds by a sequence of lemmas:

Lemma 1: If $c_1 \ge \frac{1}{n}(\sum_{i=1}^{n} c_i - M)$, then the unique solution to (1) subject to (2) and (3) is given by Q(c,M).

Proof: Denote Q(c,M) by $\overline{X}$. Clearly $\overline{X}$ satisfies (1) and (3). Now $\max_{i=1,\ldots,n}\{c_i - \overline{x}_i\} = \frac{1}{n}(\sum_{i=1}^{n} c_i - M)$.

Towards a contradiction assume that there exists $x \varepsilon \mathbf{R}^n$

satisfying (2) and (3) such that $\max_{i=1,\dots,n}\{c_i-x_i\}<\frac{1}{n}(\sum_{i=1}^{n}c_i-M)$.

But then

$$\sum_{i=1}^{n}c_i-M=\sum_{i=1}^{n}c_i-\sum_{i=1}x_i<\sum_{i=1}^{n}c_i-M$$

which is a contradiction.

Thus suppose that there exists $x\in\mathbf{R}^n, x\neq\bar{x}$, with x satisfying (2) and (3) and $\max_{i=1,\dots,n}\{c_i-x_i\}=\frac{1}{n}(\sum_{i=1}^{n}c_i-M)$.

Since $x\neq\bar{x}$ there exists j such that $c_j-x_j>\frac{1}{n}\left(\sum_{i=1}^{n}c_i-M\right)$

contradicting $\max_{i=1,\dots,n}\{c_i-x_i\}=\frac{1}{n}(\sum_{i=1}^{n}c_i-M)$..

Hence the lemma.

Q.E.D.

Note the role played by $c_1\geq\frac{1}{n}(\sum_{i=1}^{n}c_i-M)$, in the above is to ensure, $\bar{x}_i=c_i-\frac{1}{n}(\sum_{i=1}^{n}c_i-M)\geq 0\forall i$

<u>Lemma 2</u>:- Suppose $c_1<\frac{1}{n}(\sum_{i=1}^{n}c_i-M)$ and let x^* be a solution to

(1) subject to (2) and (3). Then (a) $x_1^*=0$ (b) $c_1-x_1^*< \max_{i=1,\ldots,n} \{c_i-x_i^*\}$, (c) $\sum_{i=2}^{n} c_i>M$.

<u>Proof</u> : We prove (b) first

Suppose $c_1-x_1^*= \max_{i=1,\ldots,n} \{c_i-x_i^*\}$

Then $c_1\geq c_1-x_1^*\geq c_i-x_i^*$ $\forall i$ implies $c_1\geq\frac{1}{n}\left\{\sum_{i=1}^{n} c_i-M\right\}$ which is a contradiction.

This proves (b).

Given (b) we now prove (a).

Suppose $x_1^*>0$. Clearly $x_1^*\leq c_1$. By (a) if $c_j-x_j^*= \max_{i=1,\ldots,n}\left\{c_i-x_i^*\right\}$, *then* $j\neq1$. Let $K=\left\{j/c_j-x_j^*= \max_{i=1,\ldots,n} \{c_i-x_i^*\}\right\}$

Clearly $K\neq\theta$ *and* $1\notin K$. Let $\epsilon_1>0$ be such that $c_j-x_j^*-\epsilon_1/|K| > c_i-x_i^*$ $\forall$ $j\in K, i\notin K$.

and $x_j^* + \epsilon_1/|K|< c_j$ $\forall j \in K$. Such an ϵ_1 clearly exists.

Let $\epsilon_2 > 0$ *be such that* $x_1^*- \epsilon_2 > 0$

and $c_1 - x_1^* + e_2 < c_j - x_j^* - \frac{e_1}{|K|} \quad \forall \; j \in K$

Let $e = \min \{e_1, e_2\} > 0$.

Define $x \in \mathbb{R}_+^n$ as follows :

$$x_1 = x_1^* - e$$

$$x_i = x_i^* \quad \forall i \notin K \cup \{1\}$$

$$x_i = x_i^* + \frac{e}{|K|} \quad \forall i \in K$$

Clearly $\max_{i=1,\dots,n} \{c_i - x_i\} < \max_{i=1,\dots,n} \{c_i - x_i^*\}$ Contradicting x^* solves (1) subject to (2) and (3).

Thus $x_1^* = 0$.

Finally we prove (c):

$$c_1 < \frac{1}{n}\left(\sum_{i=1}^{n} c_i - M\right)$$

$$\rightarrow n c_1 < \sum_{i=1}^{n} c_i - M$$

$$\rightarrow (n-1)\, c_1 < \sum_{i=2}^{n} c_i - M$$

Since k (c,M) > 1 and $c_1>0$, we get

$$\sum_{i=2}^{n} c_i - M > 0 .$$

In the above lemma we made use of the fact that

$\max_{i=1,\ldots,n} \{c_i - x_i\} > 0 \quad \forall x \in \mathbf{R}^{*}$ satisfying (2) and (3), in order to select an ϵ_1. This is true; for if $\max_{i=1,\ldots,n} \{c_i - x_i\} \leq 0$, then

$\sum_{i=1}^{n} c_i - M \leq o$ which contradicts that

$$c_1 < \frac{1}{n}\left(\sum_{i=1}^{n} c_i - M\right).$$

Now we proceed to prove the main theorem.

<u>Proof of main theorem</u>: If $c_1 \geq \frac{1}{n}\left(\sum_{i=1}^{n} c_i - M\right)$

then by Lemma 1 we get Q(c,M) is the unique solution to the programming problem (1) subject to (2) and (3).

If $c_1 < \frac{1}{n}\left(\sum_{i=1}^{n} c_i - M\right)$, then by lemma 2, if x^{*} is the solution to (1) subject to (2) and (3), then $x_1^{*} = 0$ *and* $(x_2^{*}, \ldots, x_n^{*})$ *solves*

$$\min_{(x_2,\ldots,x_n)} \left\{ \max_{i=2,\ldots,n} \{c_i - x_i\} \right\} \quad s.t. \quad 0 \le x_i \le c_i,\ 1=2,\ldots,n$$

$$\sum_{i=2}^{n} x_i = M$$

We are now back to an (n-1) dimensional problem for which we either apply lemma 1 or lemma 2. Proceeding thus we get that Q (c,M) is the unique solution to the programming problem (1) subject to (2) and (3).

Q.E.D.

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