



An Examination of One Dimension Marginal Distributions: Selling and Non-Selling Activities of a Salesperson

**Dheeraj Sharma
Mir Ghulam Haider Talppur**

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INDIAN INSTITUTE OF MANAGEMENT
AHMEDABAD-380 015
INDIA

An Examination of One Dimension Marginal Distributions: Selling and Non-Selling Activities of a Salesperson

Dheeraj Sharma
Indian Institute of Management, Ahmedabad
e-mail: dsharma@iima.ac.in

Mir Ghulam Haider Talpur
University of Sindh, Jamshoro

Abstract

Past researchers have endeavored to examine and ascertain the time that salespeople spend engaged in core and non-core activities. In this study, the time spent by a salesperson on non-core activities is called vacation time. This study examines the number of times a salesperson engages in vacation and the time taken by the number of vacations by controlling the number of customers. The one dimensional marginal probability generating (transform), density and cumulative distribution functions of the random variables T_{ξ_N} , ξ_N and η_N are obtained by controlling the variability of two random variables simultaneously.

KEYWORDS: Renewal process, Erlang process, Markov process, Marginal distributions

An Examination of One Dimension Marginal Distributions: Selling and Non-Selling Activities of a Salesperson

INTRODUCTION

In most retail settings, a salesperson performs numerous sales related activities, namely, selling merchandize, assorting merchandize, arranging merchandize, etc. However, the core function of a salesperson is selling merchandize. Instances are galore, when customers are seen in store waiting to be checked out. Retailers are confronted with a dilemma. On one end, retailers would prefer to keep customers waiting in queue to increase the sales in impulse buy section. On the other end, retailers would not like to keep them waiting too long for the fear of upsetting the customer. In this study, we proffer a mechanism for ascertaining the time that a salesperson spends on the core and non-core activities. Estimating the time spent on core and non-core activities will enable managers to precisely study the waiting time of the customers. Furthermore, by estimating the time spent in core activities, managers can set the work schedules of the salespeople appropriately, in turn, enabling them to best utilize the workforce.

Uchida and Aki (1995) considered that the recurrence relations of the probability generating functions (p.g.f.s) of the distributions of the sooner or later waiting time between F_0 and F_1 by the non overlapping way of counting and by the overlapping way of counting in the Markov chain. They also obtained recurrence relations of the p.g.f.s of the distributions of the sooner or later waiting time by the non overlapping way of counting of “0” runs length r or more and “1” run of length k or more in the Markov chain. The recurrence relations of the p.g.f.s of the waiting time distributions between F_0 and F_1 by the non overlapping way of counting in Markov chain was also discussed by Feller (1968). The recurrence relations of the p.g.f.s of the sooner and later waiting time distributions between F_0 and F_1 by the overlapping way of counting in Markov chain was extended from the work of King (1988). The recurrence relations of the p.g.f.s of the sooner and later waiting time distributions between F_0 and F_1 by the non overlapping way of counting of “0” runs of length r or more and “1” runs of length k or more in Markov chain was discussed in the sense of Goldstein (1990). Viveros and Balakrishnan (1993) emphasized on few applications of the geometric distribution of order k by using Bernoulli trial

with common success probability p the geometric distribution of order k was one of the simplest waiting time distribution. Several waiting time problems had been studied by many authors Li (1980), Gerber and Li (1981), Ling (1981), Aki (1981), Aki and Hirano (1993) and Fu (1986, 1995, 1996), Ebnesahrashoob and Sobel (1990), Aki (1992), Balasubramanian et al. (1993), Kutras and Papastavirdis (1993), Godbole (1993), Fu and Kutras (1994 a, 1994 b), Mohanty (1994).

The purpose was to unify various approaches which have been attempted and to extend the study of waiting time problems from the first order Markov dependent trial to the second order Markov dependent trial. The statistical analysis extended the ideas to include waiting time for the occurrence of events. This is done by replacing the Laplace transform with MGFs and incorporating probabilities into branches. Thus nodes and branches represented events and the waiting time for the occurrence of such events. The standard approach to analyze continuous time Markov chains involved solving the Chapman Kolmogrove equations for the Laplace transform of the transition probabilities or the probability generating functions of the process; it is also used by Talpur and Shi (1994).

Talpur and Shi (1994) found the one dimension marginal distributions of crossing time and renewal numbers related with two poisson processes, using probability arguments and constructing an absorbing Markov process. In this study we extend the same technique for the case of two stage Erlang process. In this study, the one dimensional marginal distributions for three random variables, namely; number of vacations, number of customer for check-out, and time spent by the salesperson in non-core activities, are obtained.

1.2 PROBLEM DESCRIPTION

After conducting a thorough inquiry into the extant literature, it can be stated that renewal processes are widely used in reliability theory and models of queuing theory. The two theories are based on counting processes. It is in common practice that one has to deal with the situations where the difference between two or more counting processes is examined. The stochastic

processes are found very helpful in analyzing such type of situations. Kroese (1992) showed the difference process of the two counting processes as

$$D(t) = N_1(t) - N_2(t)$$

Where $N_1(t)$ and $N_2(t)$ are two counting processes associated with corresponding renewal sequence of $\{X_i\}$ and $\{Y_j\}$. The problem considered for this study is extended from the work of Kroese (1992) and then Talpur and Shi (1994). It is based upon the renewal sequence of two variables $\{X_i\}$ and $\{Y_j\}$ as shown in the fig-1

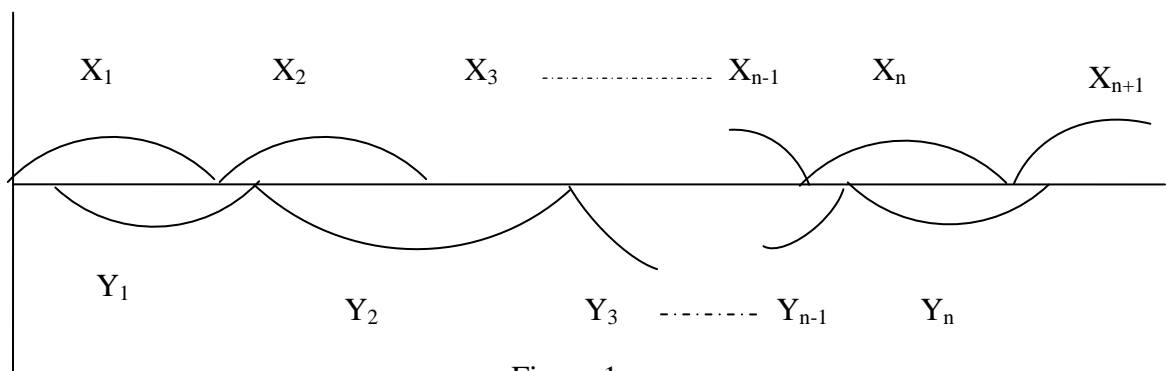


Figure 1

$$\xi_N = \min_{n \rightarrow \infty} \{n/T_n \geq S_N\}$$

Where ξ_N is random variable, N is a constant.

$$S_0 = 0, S_N = X_1 + X_2 + \dots + X_n$$

$$T_0 = 0, T_n = Y_1 + Y_2 + \dots + Y_n$$

$$T_{\xi_N} = \sum_{j=1}^{\xi_N} Y_j$$

X is representing the inter arrival, Y is the number of vacations. Both variables are discrete having renewal processes at each occurrence. The level of absorption was achieved at nth arrival

of X_n . After n th arrival the n th vacation Y_n of the core sales activity would happen. The difference of the time at which the n th vacation happened and the n th customer arrived is the crossing time. The probability generating function, Probability density function, Cumulative probability distribution function for joint distribution, for the three random variables, T_{ξ_N} , ξ_N and η_N is obtained.

1.3 ASSUMPTIONS.

Let N be a constant, $\{X_i\}$ and $\{Y_j\}$ be two sequences of random variables. Suppose that $\{X_i\}$, $i = 1, 2, 3, \dots$; independently and identically distributed with finite mean λ^{-1} and $\{Y_j\}$, $j = 1, 2, 3, \dots$; are independently and identically distributed (i.i.d) with finite mean μ^{-1}

$N_1(t)$ is the Erlang process associated with $\{X_i\}$ in which the distribution of $\{X_i\}$ is 2-stage Erlang distribution.

$N_2(t)$ is the Erlang process associated with $\{Y_j\}$ in which the distribution of $\{Y_j\}$ is 2-stage Erlang distribution.

X_i and Y_j are mutually independent.

2 ABSORBING MARKOV PROCESS AND ABSORBING TIME DISTRIBUTION

We consider a Markov process $\{X(t), t \geq 0\}$ on the state space $E = (0, 1, 2, \dots)$. If E_0 and E_1 are two non null sub set of E and they satisfy;

- 1) $E_0 \cap E_1 = \emptyset$, $E_0 \cup E_1 = E$, In this case E_0, E_1 are called a partition of E .
- 2) E_0 is the absorbing state set and E_1 is the transient state set.
- 3) for given initial condition α_E

The absorbing Markov process (A.M.P) is constructed to analyze the problem consider the AMP $\{N_1(t), N_2(t), I(t), J(t)\}$ in which $N_1(t)$, and $N_2(t)$ are the counting process associated with X_i and Y_j respectively.

$$E = \{ (i,k,j,l), (i',j') / i, j = 0,1, \dots; k, l = 1,2; i' = N', N'+1', \dots; j' = 1',2', \dots \}$$

Where (i', j') are absorbing states. Transitions of states are shown in the figure 2.

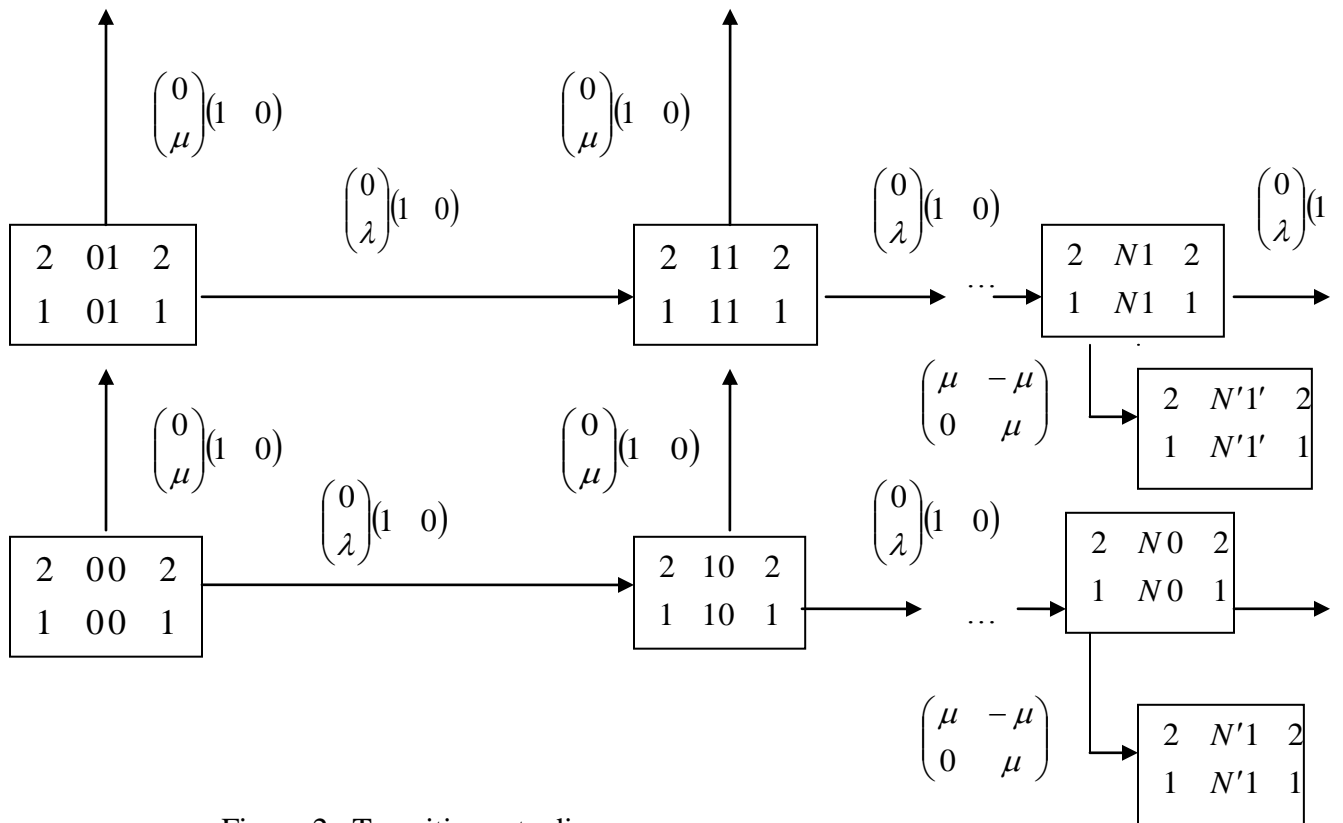


Figure 2. Transition rate diagram

Let

$$P_{ij}(k, l, t) = p\{N_1(t) = i, N_2(t) = j, I(t) = k, J(t) = l\}$$

and

$$P_{ij}(t) = [p_{ij}(1,1,t), \dots, p_{ij}(1,n,t), \dots, p_{ij}(m,1,t), \dots, p_{ij}(m,n,t)]$$

By the transition-rate diagram we can get the system of differential equations as follows

$$P'_{ij}(t) = P_{ij}(t) \left\{ - \left\{ \begin{pmatrix} \lambda & -\lambda \\ 0 & \lambda \end{pmatrix} + \begin{pmatrix} \mu & -\mu \\ 0 & \mu \end{pmatrix} \right\} \right\} + P_{i-1,j}(t) \begin{pmatrix} 0 & 0 \\ \lambda & 0 \end{pmatrix} + P_{ij-1}(t) \begin{pmatrix} 0 \\ \mu \end{pmatrix} \begin{pmatrix} 1 & 0 \end{pmatrix}$$

$$i = 0, 1, \dots, N-1; j = 0, 1, 2, \dots; \quad (2.1)$$

$$P'_{ij}(t) = P_{ij}(t) \left\{ - \left\{ \begin{pmatrix} \lambda & -\lambda \\ 0 & \lambda \end{pmatrix} + \begin{pmatrix} \mu & -\mu \\ 0 & \mu \end{pmatrix} \right\} \right\} + P_{i-1,j}(t) \begin{pmatrix} 0 & 0 \\ \lambda & 0 \end{pmatrix}, i = N, N+1, \dots; j = 0, 1, \dots; \quad (2.2)$$

From these differential difference equations we have obtained the joint distribution for two and three random variables in our separate paper. The one dimension marginal distributions for the same case are obtained in this study.

3. METHODOLOGY

The one dimensional marginal probability generating functions (probability transform functions), one dimensional probability density functions and cumulative probability distribution functions for random variables T_{ξ_N} , ξ_N and η_N are obtained by controlling the variability of two random variables simultaneously and find the effect of individual variable at one time.

3.1 PROBABILITY GENERATING FUNCTION FOR T_{ξ_N}

Theorem 3.1.1; The one dimensional probability transform function of the random variable T_{ξ_N}

$$f^*(s) = (1 \ 0) \left\{ \begin{pmatrix} s + \lambda + \mu & -(\lambda + \mu) \\ -\mu & s + \lambda + \mu \end{pmatrix}^{-1} \begin{pmatrix} 0 & 0 \\ \lambda & 0 \end{pmatrix} \right\}^N \begin{pmatrix} s + \lambda + \mu & -(\lambda + \mu) \\ -\lambda & s + \lambda + \mu \end{pmatrix}^{-1} \begin{pmatrix} \mu \\ \mu \end{pmatrix}$$

Proof; The one dimensional marginal probability generating function (transform function) for the random variable T_{ξ_N} is computed from the joint probability generating function of three random variables T_{ξ_N} , ξ_N and η_N see Iffat (2004).

$$f^*(s, u, z) = u(1 \ 0) \left\{ \begin{pmatrix} s + \lambda + \mu & -(\lambda + \mu) \\ -u\mu & S + \lambda + \mu \end{pmatrix}^{-1} \begin{pmatrix} 0 & 0 \\ \lambda z & 0 \end{pmatrix} \right\}^N \begin{pmatrix} s + \lambda + \mu & -(\lambda + \mu) \\ -\lambda z & s + \lambda + \mu \end{pmatrix}^{-1} \begin{pmatrix} \mu \\ \mu \end{pmatrix}$$

The effect of number of vacations made by a salesperson at service channels and the number of arriving customers are controlled to get the probability generating effect for the time taken by the number of vacations made by salesperson. So let z and u close to 1 to find the one dimensional marginal probability generating function for the random variable T_{ξ_N} .

$$f^*(s) = (1 \ 0) \left\{ \begin{pmatrix} s + \lambda + \mu & -(\lambda + \mu) \\ -\mu & S + \lambda + \mu \end{pmatrix}^{-1} \begin{pmatrix} 0 & 0 \\ \lambda & 0 \end{pmatrix} \right\}^N \begin{pmatrix} s + \lambda + \mu & -(\lambda + \mu) \\ -\lambda & s + \lambda + \mu \end{pmatrix}^{-1} \begin{pmatrix} \mu \\ \mu \end{pmatrix} \quad (3.1.1)$$

Theorem 3.1.2; The one dimensional marginal probability density function of random variable T_{ξ_N} .

$$P\{T_{\xi_N} \leq t\} = \binom{N+j-2}{j-1} \mu^j \lambda^i \left[\frac{(\lambda + \mu)^{j+i-1} t^{2j+2i-2}}{(2j+2i-2)!} + \frac{(\lambda + \mu)^{j+i} t^{2j+2i-1}}{(2j+2i-1)!} \right] e^{-(\lambda+\mu)t}$$

Proof; The definition of L transform can be expressed by the following equation as shown by Pipes (1970)

$$f^*(s) = \int_0^{\infty} \exp(-st) dp\{T_{\xi_N} \leq t\} \quad (3.1.2)$$

The value of $f^*(s)$ from equation (3.1.1) is placed in equation (3.1.2) yields

$$\int_0^{\infty} \exp(-st) dp\{T_{\xi_N} \leq t\} = (1 \ 0) \left\{ \begin{pmatrix} s + \lambda + \mu & -(\lambda + \mu) \\ -\mu & S + \lambda + \mu \end{pmatrix}^{-1} \begin{pmatrix} 0 & 0 \\ \lambda & 0 \end{pmatrix} \right\}^N \begin{pmatrix} s + \lambda + \mu & -(\lambda + \mu) \\ -\lambda & s + \lambda + \mu \end{pmatrix}^{-1} \begin{pmatrix} \mu \\ \mu \end{pmatrix}$$

Let $a = s + \lambda + \mu$ and multiplying and dividing by a we get

$$f^*(s) = (1 \ 0) \left\{ \left[I - \begin{pmatrix} 0 & (\lambda + \mu) \\ \frac{\mu}{a} & 0 \end{pmatrix} \right]^{-1} \begin{pmatrix} 0 & 0 \\ \frac{\lambda}{a} & 0 \end{pmatrix} \right\}^N \frac{1}{a} \left\{ I - \begin{pmatrix} 0 & (\lambda + \mu) \\ \frac{\lambda}{a} & 0 \end{pmatrix} \right\}^{-1} \begin{pmatrix} \mu \\ \mu \end{pmatrix}$$

The rule of power series is applied as in Pipes and Harwil (1970) and Talpur and Shi (1994).

$$f^*(s) = \frac{1}{a} (1 \ 0) \left\{ \sum_{k=0}^{\infty} \begin{pmatrix} 0 & (\lambda + \mu) \\ \frac{\mu}{a} & 0 \end{pmatrix}^k \begin{pmatrix} 0 & 0 \\ \frac{\lambda}{a} & 0 \end{pmatrix} \right\}^N \left\{ \sum_{l=0}^{\infty} \begin{pmatrix} 0 & (\lambda + \mu) \\ \frac{\lambda}{a} & 0 \end{pmatrix}^l \right\} \begin{pmatrix} \mu \\ \mu \end{pmatrix} \tag{3.1.3}$$

Putting these results of two set of series $\left\{ \sum_{k=0}^{\infty} \begin{pmatrix} 0 & (\lambda + \mu) \\ \frac{\mu}{a} & 0 \end{pmatrix}^k \begin{pmatrix} 0 & 0 \\ \frac{\lambda}{a} & 0 \end{pmatrix} \right\}^N$ and

$\left\{ \sum_{l=0}^{\infty} \begin{pmatrix} 0 & (\lambda + \mu) \\ \frac{\lambda}{a} & 0 \end{pmatrix}^l \right\}$ is placed in equation (2.1.3).

$$f^*(s) = \frac{1}{a} (1 \ 0) \begin{pmatrix} \left\{ \frac{\lambda}{a} \sum_{k=0}^{\infty} \left(\frac{\mu}{a} \right)^k \left(\frac{\lambda + \mu}{a} \right)^{k+1} \right\}^N & 0 \\ \left\{ \frac{\lambda}{a} \sum_{k=0}^{\infty} \left(\frac{\mu}{a} \right)^k \left(\frac{\lambda + \mu}{a} \right)^k \right\}^N \left(\frac{\lambda + \mu}{a} \right)^{N-1} & 0 \end{pmatrix} \begin{pmatrix} \sum_{l=0}^{\infty} \left(\frac{\lambda}{a} \right)^l \left(\frac{\lambda + \mu}{a} \right)^l & \sum_{l=0}^{\infty} \left(\frac{\lambda}{a} \right)^l \left(\frac{\lambda + \mu}{a} \right)^{l+1} \\ \sum_{l=0}^{\infty} \left(\frac{\lambda}{a} \right)^{l+1} \left(\frac{\lambda + \mu}{a} \right)^l & \sum_{l=0}^{\infty} \left(\frac{\lambda}{a} \right)^l \left(\frac{\lambda + \mu}{a} \right)^l \end{pmatrix} \begin{pmatrix} \mu \\ \mu \end{pmatrix}$$

After some algebraic manipulations with the application of the negative binomial distribution we obtain the following expression, as Bailey (1964).

$$f^*(s) = \sum_{k=0}^{\infty} \sum_{l=0}^{\infty} \binom{N+k-1}{k} \left(\frac{\mu}{a} \right)^{k+1} \left(\frac{\lambda}{a} \right)^{l+N} \left(\frac{\lambda + \mu}{a} \right)^{k+l+N} \left\{ 1 + \left(\frac{\lambda + \mu}{a} \right) \right\}$$

Let $j = k + 1$ and $i = l + N$, and substituting the value of a we get

$$f^*(s) = \sum_{j=1}^{\infty} \sum_{i=N}^{\infty} \binom{N+j-2}{j-1} \left(\frac{\mu}{s+\lambda+\mu} \right)^j \left(\frac{\lambda}{s+\lambda+\mu} \right)^i \left\{ \left(\frac{\lambda+\mu}{s+\lambda+\mu} \right)^{j+i-1} + \left(\frac{\lambda+\mu}{s+\lambda+\mu} \right)^{j+i} \right\}$$

Taking inverse of Laplace transform one can obtain the probability density function for continuous random variable as performed by Kreyszig (1999).

$$dp\{T_{\xi_N} \leq t\} = \binom{N+j-2}{j-1} \frac{\mu^j \lambda^i (\lambda+\mu)^{j+i-1} t^{2j+2i-2}}{(2j+2i-2)!} e^{-(\lambda+\mu)t} dt + \frac{\mu^j \lambda^i (\lambda+\mu)^{j+i} t^{2j+2i-1}}{(2j+2i-1)!} e^{-(\lambda+\mu)t} dt$$

The one dimension marginal probability density function of random variable T_{ξ_N} time taken by vacations of salesperson is obtained as

$$p\{T_{\xi_N} \leq t\} = \binom{N+j-2}{j-1} \mu^j \lambda^i \left[\frac{(\lambda+\mu)^{j+i-1} t^{2j+2i-2}}{(2j+2i-2)!} + \frac{(\lambda+\mu)^{j+i} t^{2j+2i-1}}{(2j+2i-1)!} \right] e^{-(\lambda+\mu)t} \quad (3.1.4)$$

Theorem 3.1.3; The one dimensional marginal cumulative probability distribution function of the random variable T_{ξ_N} .

$$p\{T_{\xi_N} \leq t\} = \binom{N+j-2}{j-1} \left(\frac{\mu^j \lambda^i}{(\lambda+\mu)^{j+i}} \right) \left\{ \sum_{r=0}^{(2j+2i-N-1)} \frac{[(\lambda+\mu)t]^r}{r!} + \sum_{r=0}^{(2j+2i-N)} \frac{[(\lambda+\mu)t]^r}{r!} \right\} e^{-(\lambda+\mu)t} \quad (3.1.5)$$

Proof; The cumulative probability function can be defined as

$$p\{T_{\xi_N} \leq t\} = \int_t^{\infty} p\{T_{\xi_N} \leq t\} dt$$

Substituting the value of $p\{T_{\xi_N} \leq t\}$ from equation no.(2.1.4) we get

$$p\{T_{\xi_N} \leq t\} = \int_t^{\infty} \binom{N+j-2}{j-1} \mu^j \lambda^i \left[\frac{(\lambda+\mu)^{j+i-1} t^{2j+2i-2}}{(2j+2i-2)!} + \frac{(\lambda+\mu)^{j+i} t^{2j+2i-1}}{(2j+2i-1)!} \right] e^{-(\lambda+\mu)t} dt$$

Integration by parts is used for finding the cumulative probability distribution function of time spent in vacations taken by salesperson as done by Medhi (1982), so the one dimensional cumulative distribution function for the random variable T_{ξ_N} is established.

3.2 One Dimensional Marginal Probability Distribution Functions For η_N

The effect of number of arriving customers represented by η_N is studied by controlling the time taken by the number of vacations made and the number of vacations made by different service channels.

Theorem 3.2.1; The one dimensional probability generating function (probability transform function) for the random variable η_N .

$$f(z) = (1 \ 0) \left\{ \begin{pmatrix} \lambda + \mu & -(\lambda + \mu) \\ -\mu & \lambda + \mu \end{pmatrix}^{-1} \begin{pmatrix} 0 & 0 \\ z\lambda & 0 \end{pmatrix} \right\}^N \begin{pmatrix} \lambda + \mu & -(\lambda + \mu) \\ -z\lambda & \lambda + \mu \end{pmatrix}^{-1} \begin{pmatrix} \mu \\ \mu \end{pmatrix}$$

Proof; The one dimensional probability generating function (probability transform function) for the random variable η_N is obtained from the joint probability generating function for three random variables T_{ξ_N} , ξ_N and η_N see Iffat (2004).

$$f^*(s, u, z) = u(1 \ 0) \left\{ \begin{pmatrix} s + \lambda + \mu & -(\lambda + \mu) \\ -u\mu & s + \lambda + \mu \end{pmatrix}^{-1} \begin{pmatrix} 0 & 0 \\ \lambda z & 0 \end{pmatrix} \right\}^N \begin{pmatrix} s + \lambda + \mu & -(\lambda + \mu) \\ -\lambda z & s + \lambda + \mu \end{pmatrix}^{-1} \begin{pmatrix} \mu \\ \mu \end{pmatrix}$$

The effect of the time taken by the number of vacations made and that of the random variable number of vacations made are controlled by putting s and u close to 0 and 1 respectively. The one dimensional marginal probability generating function is computed as

$$f(z) = (1 \ 0) \left\{ \begin{pmatrix} \lambda + \mu & -(\lambda + \mu) \\ -\mu & \lambda + \mu \end{pmatrix}^{-1} \begin{pmatrix} 0 & 0 \\ z\lambda & 0 \end{pmatrix} \right\}^N \begin{pmatrix} \lambda + \mu & -(\lambda + \mu) \\ -z\lambda & \lambda + \mu \end{pmatrix}^{-1} \begin{pmatrix} \mu \\ \mu \end{pmatrix} \quad (3.2.1)$$

Theorem 3.2.2; The one dimensional probability density function for the random variable η_N .

$$p\{\eta_N = i\} = \sum_{j=1}^{\infty} \binom{N+j-2}{j-1} 2 \left(\frac{\mu}{\lambda + \mu} \right)^j \left(\frac{\lambda}{\lambda + \mu} \right)^i$$

Proof; The definition of z transform can be expressed by the following equation as given by Talpur and Shi (1994).

$$f(z) = \sum_{i=N}^{\infty} p\{\eta_N = i\} z^i$$

So by putting the value of $f(z)$ from the equation No. (3.2.1) we yields

$$\sum_{i=N}^{\infty} p\{\eta_N = i\}z^i = (1 \ 0) \left\{ \begin{pmatrix} \lambda + \mu & -(\lambda + \mu) \\ -\mu & \lambda + \mu \end{pmatrix}^{-1} \begin{pmatrix} 0 & 0 \\ z\lambda & 0 \end{pmatrix} \right\}^N \begin{pmatrix} \lambda + \mu & -(\lambda + \mu) \\ -z\lambda & \lambda + \mu \end{pmatrix}^{-1} \begin{pmatrix} \mu \\ \mu \end{pmatrix}$$

Let $a = \lambda + \mu$, and dividing by a one can get

$$\begin{aligned} f(z) &= (1 \ 0) \left\{ \begin{pmatrix} 1 & -1 \\ -\frac{\mu}{a} & 1 \end{pmatrix}^{-1} \begin{pmatrix} 0 & 0 \\ \frac{\lambda z}{a} & 0 \end{pmatrix} \right\}^N \frac{1}{a} \begin{pmatrix} 1 & -1 \\ -\frac{z\lambda}{a} & 1 \end{pmatrix}^{-1} \begin{pmatrix} \mu \\ \mu \end{pmatrix} \\ &= (1 \ 0) \left\{ I - \begin{pmatrix} 0 & 1 \\ \frac{\mu}{a} & 0 \end{pmatrix} \begin{pmatrix} 0 & 0 \\ \frac{\lambda z}{a} & 0 \end{pmatrix} \right\}^N \frac{1}{a} \left\{ I - \begin{pmatrix} 0 & 1 \\ \frac{z\lambda}{a} & 0 \end{pmatrix} \right\}^{-1} \begin{pmatrix} \mu \\ \mu \end{pmatrix} \end{aligned}$$

The rule of geometric series is applied as expressed by saaty (1961).

$$f(z) = (1 \ 0) \left\{ \sum_{k=0}^{\infty} \begin{pmatrix} 0 & 1 \\ \frac{\mu}{a} & 0 \end{pmatrix}^k \begin{pmatrix} 0 & 0 \\ \frac{\lambda z}{a} & 0 \end{pmatrix} \right\}^N \frac{1}{a} \left\{ \sum_{l=0}^{\infty} \begin{pmatrix} 0 & 1 \\ \frac{z\lambda}{a} & 0 \end{pmatrix}^l \right\} \begin{pmatrix} \mu \\ \mu \end{pmatrix} \tag{3.2.2}$$

The results of two set of series $\left\{ \sum_{k=0}^{\infty} \begin{pmatrix} 0 & 1 \\ \frac{\mu}{a} & 0 \end{pmatrix}^k \begin{pmatrix} 0 & 0 \\ \frac{\lambda z}{a} & 0 \end{pmatrix} \right\}^N$ and $\left\{ \sum_{l=0}^{\infty} \begin{pmatrix} 0 & 1 \\ \frac{z\lambda}{a} & 0 \end{pmatrix}^l \right\}$ is placed in

equation (3.2.2) and after some simplification we gets

$$f(z) = \frac{1}{a} \left\{ \frac{\lambda z}{a} \sum_{k=0}^{\infty} \left(\frac{\mu}{a} \right)^k \right\}^N \left\{ \mu \sum_{l=0}^{\infty} \left(\frac{\lambda z}{a} \right)^l + \mu \sum_{l=0}^{\infty} \left(\frac{\lambda z}{a} \right)^l \right\}$$

Using applications of the negative binomial distribution as Hogg and Craig (1995) one can obtained.

$$f(z) = \sum_{k=0}^{\infty} \sum_{l=0}^{\infty} \binom{N+k-1}{k} 2 \left(\frac{\mu}{a} \right)^{k+1} \left(\frac{\lambda}{a} \right)^{l+N} z^{l+N}$$

By taking $j = k + 1$ and $i = l + N$, and substituting the value of a and comparing the coefficient of z we yields

$$p\{\eta_N = i\} = \sum_{j=1}^{\infty} \binom{N+j-2}{j-1} 2 \left(\frac{\mu}{\lambda + \mu} \right)^j \left(\frac{\lambda}{\lambda + \mu} \right)^i \tag{3.2.3}$$

This is looks like as a negative binomial distribution, which is the convolution of the geometric distribution as expressed by Feller (1970). The vacation will vanish at this stage, as the crossing

time is attained by reaching the absorbing state. The above equation calculates the number of customers in a given setting.

Theorem 3.2.3; The one dimensional cumulative probability distribution function for the random variable η_N .

$$P\{\eta_N = i\} = \sum_{j=1}^{\infty} \sum_{i=N}^{\infty} \binom{N+j-2}{j-1} 2 \left(\frac{\mu}{\lambda + \mu} \right)^j \left(\frac{\lambda}{\lambda + \mu} \right)^i \quad (3.2.4)$$

Proof; The one dimensional cumulative probability distribution function for random variable η_N is obtained by summing the one dimensional marginal probability density function for the discrete random variable η_N the number of arriving customers from equation (3.2.3) the proof is obvious.

The cdf for the number of customers arrived act as negative binomial distribution, where the number of vacations considered until crossing time for having the absorbing state and the number of customers are taken after the achievement of absorption state.

3.3 One Dimensional Marginal Probability Distribution Functions For ξ_N .

The effect of number of vacations made by service channels represented by ξ_N is studied by controlling the time taken by the number of vacations made and the number of arriving customers or units. The one dimensional marginal probability generating function (probability transform function), density function and cumulative distribution function for the random variable ξ_N .

Theorem 3.3.1; The one dimensional marginal probability generating function (probability transform function) for the random variable ξ_N is

$$f(u) = u(1 \ 0) \left\{ \begin{pmatrix} \lambda + \mu & -(\lambda + \mu) \\ -u\mu & \lambda + \mu \end{pmatrix}^{-1} \begin{pmatrix} 0 & 0 \\ \lambda & 0 \end{pmatrix} \right\}^N \begin{pmatrix} \lambda + \mu & -(\lambda + \mu) \\ -\lambda & \lambda + \mu \end{pmatrix}^{-1} \begin{pmatrix} \mu \\ \mu \end{pmatrix} \quad (3.3.1)$$

Proof; The one dimensional probability generating function for the random variable ξ_N is obtained from the joint probability generating function for three random variables T_{ξ_N} , ξ_N and η_N see Iffat (2004).

$$f^*(s, u, z) = u(1 \ 0) \left\{ \begin{pmatrix} s + \lambda + \mu & -(\lambda + \mu) \\ -u\mu & s + \lambda + \mu \end{pmatrix}^{-1} \begin{pmatrix} 0 & 0 \\ \lambda z & 0 \end{pmatrix} \right\}^N \begin{pmatrix} s + \lambda + \mu & -(\lambda + \mu) \\ -\lambda z & s + \lambda + \mu \end{pmatrix}^{-1} \begin{pmatrix} \mu \\ \mu \end{pmatrix}$$

The effect of the random variable T_{ξ_N} time taken by the number of vacations made and that of the random variable η_N number of customers arrived are controlled by putting s and z close to 0 and 1 respectively the proof is obvious.

Theorem 3.3.2; The one dimensional marginal probability density function (pdf) for the random variable ξ_N is

$$p\{\xi_N = j\} = \sum_{i=N}^{\infty} \binom{N+j-2}{j-1} 2 \left(\frac{\mu}{\lambda + \mu} \right)^j \left(\frac{\lambda}{\lambda + \mu} \right)^i$$

Proof; The definition of z transform can be expressed by the following equation as given by Talpur and Shi (1994).

$$f(u) = \sum_{j=1}^{\infty} p\{\xi_N = j\} u^j$$

So by putting the value of $f(u)$ from equation (3.3.1)

$$f(u) = \sum_{j=1}^{\infty} p\{\xi_N = j\} u^j = u(1 \ 0) \left\{ \begin{pmatrix} \lambda + \mu & -(\lambda + \mu) \\ -u\mu & \lambda + \mu \end{pmatrix}^{-1} \begin{pmatrix} 0 & 0 \\ \lambda & 0 \end{pmatrix} \right\}^N \begin{pmatrix} \lambda + \mu & -(\lambda + \mu) \\ -\lambda & \lambda + \mu \end{pmatrix}^{-1} \begin{pmatrix} \mu \\ \mu \end{pmatrix}$$

Taking $a = s + \lambda + \mu$, and dividing by a one can get

$$f(u) = \frac{u}{a} (1 \ 0) \left\{ I - \begin{pmatrix} 0 & 1 \\ \frac{u\mu}{a} & 0 \end{pmatrix}^{-1} \begin{pmatrix} 0 & 0 \\ \frac{\lambda}{a} & 0 \end{pmatrix} \right\}^N \left\{ I - \begin{pmatrix} 0 & 1 \\ \frac{\lambda}{a} & 0 \end{pmatrix}^{-1} \right\} \begin{pmatrix} \mu \\ \mu \end{pmatrix}$$

Applying rule of the geometric series it can be expressed as

$$f(u) = \frac{u}{a} (1 \ 0) \left\{ \sum_{k=0}^{\infty} \begin{pmatrix} 0 & 1 \\ \frac{u\mu}{a} & 0 \end{pmatrix}^k \begin{pmatrix} 0 & 0 \\ \frac{\lambda}{a} & 0 \end{pmatrix} \right\}^N \left\{ \sum_{l=0}^{\infty} \begin{pmatrix} 0 & 1 \\ \frac{\lambda}{a} & 0 \end{pmatrix}^l \right\} \begin{pmatrix} \mu \\ \mu \end{pmatrix}$$

By simplifying this and applying negative binomial distribution as done by Saaty (1961), Hogg and Craig (1995) we get

$$f(u) = \sum_{k=0}^{\infty} \sum_{l=0}^{\infty} \binom{N+k-1}{k} 2 \left(\frac{\mu}{a} \right)^{k+1} \left(\frac{\lambda}{a} \right)^{l+N} u^{k+1}$$

By putting $j = k + 1$ and $i = l + N$ and substituting the value of a and comparing the coefficient of u we obtained the proof.

$$P\{\xi_N = j\} = \sum_{i=N}^{\infty} \binom{N+j-2}{j-1} 2 \left(\frac{\mu}{\lambda + \mu} \right)^j \left(\frac{\lambda}{\lambda + \mu} \right)^i \quad (3.3.2)$$

The pdf for the discrete random variable of the number of vacations performed by service channels until having the absorbing state is expressed as a negative binomial distribution.

Theorem 3.3.3; The one dimensional marginal cumulative probability distribution function for the random variable ξ_N .

$$P\{\xi_N = j\} = \sum_{j=1}^{\infty} \sum_{i=N}^{\infty} \binom{N+j-2}{j-1} 2 \left(\frac{\mu}{\lambda + \mu} \right)^j \left(\frac{\lambda}{\lambda + \mu} \right)^i$$

Proof; The one dimensional cumulative probability distribution function for random variable ξ_N is obtained by summing the one dimensional marginal probability density function for the discrete random variable ξ_N the number of vacations made by service channels from equation (3.3.2).

$$P\{\xi_N = j\} = \sum_{i=N}^{\infty} \binom{N+j-2}{j-1} 2 \left(\frac{\mu}{\lambda + \mu} \right)^j \left(\frac{\lambda}{\lambda + \mu} \right)^i$$

The one dimensional cumulative probability distribution function (cdf) for the random variable ξ_N .

$$P\{\xi_N = j\} = \sum_{j=1}^{\infty} \sum_{i=N}^{\infty} \binom{N+j-2}{j-1} 2 \left(\frac{\mu}{\lambda + \mu} \right)^j \left(\frac{\lambda}{\lambda + \mu} \right)^i \quad (3.3.3)$$

The cdf for the number of vacations customers arrived accomplishes the negative binomial distribution, where the observed number of vacations made by a salesperson is related to the crossing time for having the absorption state and the number of customers is taken after the achievement of absorption state.

4. RESULTS AND DISCUSSION

Above theorems expressed that the probability density functions are related to discrete random variables and could be linked to the Poisson process as shown by Medhi (1982). The crossing time shows the two stage Erlang distribution, the number of vacations related to crossing time follows the negative binomial distribution and the number of arrivals for the absorption state also satisfies the negative binomial distribution. The results of this study allow us the derivation and calculation of several performance measures including the average lag time between customers, the expected number customers, and the probability of encountering situations such as free time and busy time.

In this study, we have provided a mechanism to estimate probability distribution number of customers, number of vacations, and time spent on vacations by a salesperson. The results of this study are in concurrence with the Erlang distribution. Hence, results can be used to measure the time between two customers. Estimation of time between customers can be used in conjunction with the expected time the salesperson spends on non-selling functions of their jobs (e.g. filling sales call reports, scheduling appointment, etc.), and hence, to estimate the time customer focuses on selling activities.

Sales managers can utilize the information in developing schedules for the salespeople to obtain optimal performance outcomes. Estimation of average time spent on selling activities, non-selling activities, and average time between customers can aid sales managers to appropriately develop quotas for salespeople. Furthermore, sales managers can select the right number of workforce for performing sales function in their organization. In other words, sales managers can have a better understanding of job description of the salespeople. By estimation of time spent on selling and non-selling activities, the sales managers can appropriately adjust the sales quotas of the salespeople because salespeople may not be spending all their time performing selling functions. Additionally, information about the waiting time can allow the sales managers to utilize the salespeople to work on non-selling function during the waiting time. In essence, more efficient scheduling and optimal quotas can be developed by sales manager, which in turn, will lead to optimal performance outcomes.

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