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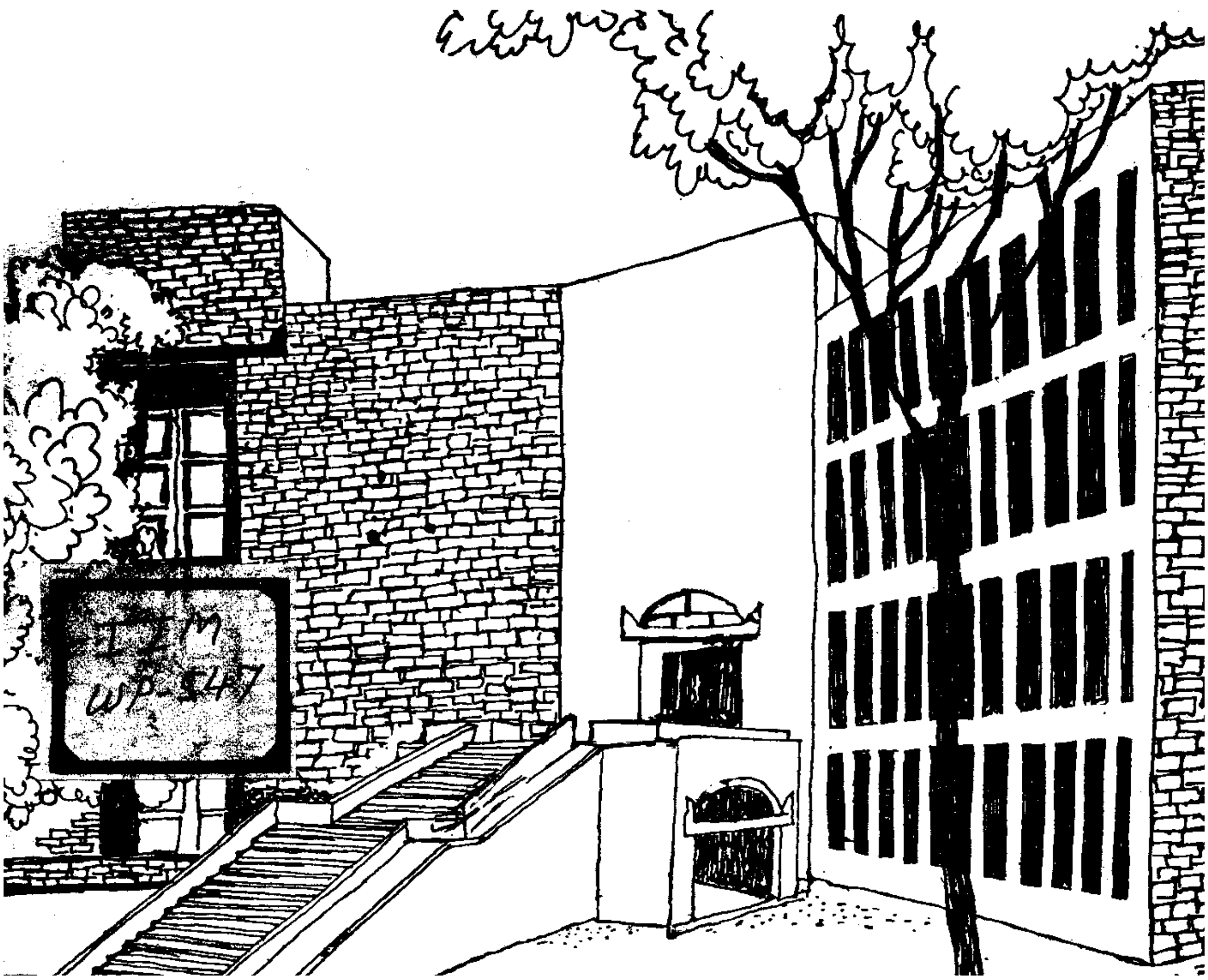

ON THE ESTIMATION OF ELASTICITY IN ECONOMICS

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ABSTRACT

For the measurement of elasticity, given two observations of a bivariate relationship, the arc elasticity formula has been traditionally used by economists and statisticians. However, no proper statistical justification for this procedure exists in the literature. In this paper measures of elasticity on an arc are derived using widely accepted statistical criteria such as the minimum absolute deviation criterion and the least squares criterion. It is shown that in the linear bivariate case the minimum absolute deviation elasticity is the arc elasticity. However, the formulae according to the least squares criterion and other criteria differ from arc elasticity even in the linear case. A numerical comparision of formulae is also provided and these are assessed on the basis of a'goodness of fit' statistic developed for this purpose,

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1. Introduction

The concept of elasticity is widely used in economics for quantifying the degree of responsiveness in different contexts. Two alternative measures have usually been employed for this purpose, namely, point elasticity and arc elasticity. Point elasticity, E(p), at point (p,q) in two dimensional space is defined in context to a function

$$q = f(\cdot p,...) \qquad (1.1)$$

as follows:

$$E(p) = \frac{\delta \log q}{\delta \log p} = \frac{\delta p}{\delta q} \cdot \frac{q}{p} \qquad (1.2)$$

where the partial derivative is used instead of the total derivative because the function (1.1) may involve several other variables besides p. The function (1.1) may be interpreted as demand function when q stands for demand p stands for price. It may in fact represent any other function depending upon the meaning assigned to q and p in proper context. Arc elasticity, E_A, is defined over an interval say $[(f_0,q_0), (p_1,q_1)]$ in two dimensional spaces for two observation points as follows:

$$E_A = \frac{q_1 - q_0}{p_1 - p_0} \cdot \frac{p_1 + p_0}{q_1 + q_0} \qquad (1.3)$$

Implicitly this measure of elasticity is supposed to hold good for any functional form between q and p. This is the only interpretation that can be made in the presence of complete silence in the literature relating to the shape of the curve between the points (p_0, q_0) and (p_1, q_1). Thus arc elasticity, E_A, may be supposed to be defined for an interval corresponding to function (1.1).

It is well known that the elasticity coefficient E(p), as in (1.2), remains invariant when the function (1.1) is of double log form. It changes for each point of the function for any other functional form. Therefore, if the functional form of (1.1) is not a double log one, the point elasticity, E(p), will vary for each point within the interval $[(p_0, q_0), (p_1, q_1)]$ and we have a set of point elasticities like (1.2). The question that arises is what criterion of estimation is satisfied implicitly when we use arc elasticity in such a case. In fact, the problem is a more general one. One may like to find a proper estimate of elasticity coefficients in such a situation.

This question is attempted to be answered in this paper. We develop estimators of elasticity over an arc in accordance with a number of possible statistical approaches. Section II contains derivation of such estimators along with their special cases. Section III provides illustrative examples of numerical comparisons. The last section contains concluding remarks.

2. Alternative Estimators of Elasticity

The following criteria, all of which are to be discussed in this section, are widely used in statistics in the design of estimators:

a) The least squares (LS) criterion.

b) The minimum absolute deviation (MAD) criterion.

c) The minimum maximum deviation (MMX) criterion.

d) The Average (AV) criterion.

Furthermore, we compare these estimators with

e) the arc elasticity estimator; and

f) the constant elasticity estimator.

We assume throughout that E(p) is continuous.

a. The least-squares elasticity estimator (E_L) and the average elasticity estimator

The least squares elasticity estimator is found by solving the following problem (it is assumed without loss of generality that p_1 is greater than p_0):

$$\min_{E_L} \int_{p_0}^{p_1} (E(p) - E_L)^2 \, dp \qquad (2.1)$$

(2.1) may be expanded to get the following expression:

$$\min_{E_L} \left(\int_{p_0}^{p_1} E(p)^2 \, dp + \int_{p_0}^{p_1} E_L{}^2 \, dp - \int_{p_0}^{p_1} 2E(p)E_L \, dp \right) \qquad (2.2)$$

If we denote the first integral in (2.2) by A (noting that it is independant of E_L) we get

$$\min_{E_L} \left[A + E_L^2 (p_1 - p_0) - 2 E_L \int_{p_0}^{p_1} E(p)dp \right] \tag{2.3}$$

Differentiation of the expression in square brackets and setting it equal to zero, we get

$$2 E_L (p_1 - p_0) - 2 \int_{p_0}^{p_1} E(p)dp = 0 \tag{2.4}$$

he solution of (2.4) gives us the least squares estimator of E_L since the second order expression $2(p_1-p_0)$ is positive.

We thus have the following result.

<u>Theorem 1</u>

$$E_L = \frac{1}{(p_1-p_0)} \int_{p_0}^{p_1} E(p)dp.$$

<u>Proof</u>: See above.

Thus the least squares estimator is seen to be the average elasticity estimator.

We now consider three important special cases of the function f(p,.) and compute the elasticity estimators in these three cases.

(i) The linear case: $q = a+bp$; $a \neq 0$; $b \neq 0$; (2.5)

(ii) The semi-log case: $\ln q = a+bp$; $a \neq 0$; $b \neq 0$; (2.6)

(iii) The double-log case: $\ln q = a+b \ln p$; $a \neq 0$; $b \neq 0$ (2.7)

The average elasticities in these cases are given in the following theorem.

<u>Theorem 2</u>

Given $p_1 > p_2$; $a \neq 0$ and $b \neq 0$ and two observations of a function $[(p_0, q_0), (p_1, q_1)]$,

(i) $E_L = 1 - \frac{q_0 p_1 - q_1 p_0}{(q_1 - q_0)(p_1 - p_0)} \ln (q_1/q_0)$ in the linear case,

(ii) $E_L = \ln (q_1/q_0)(p_1+p_0) / 2(p_1-p_0)$ in the semi-log case.

(iii) $E_L = b = \ln (q_1/q_0) / \ln (p_1/p_0)$ in the double-log case.

<u>Proof</u> (i) If $q = a+bp$ then $E(p) = \frac{dq}{dp} \cdot \frac{p}{q} = \frac{bp}{a+bp} = 1 - \frac{a}{a+bp}$

now
$$E_L = \frac{1}{p_1-p_0} \int_{p_0}^{p_1} E(p)\, dp$$

$$= \frac{1}{p_1-p_0} \int_{p_0}^{p_1} \left(1 - \frac{a}{a+bp}\right) dp$$

$$= \frac{1}{p_1-p_0} \quad (p_1 - p_0) - \frac{a}{b}(\ln q_1 - \ln q_o)$$ when we have used $a+bp_i = q_i$.

Now since $b = \frac{q_1 - q_0}{p_1 - p_0}$ and $a = \frac{p_1 q_0 - p_0 q_1}{p_1 - p_0}$,

we get $$E_L = 1 - \frac{p_1 q_0 - p_0 q_1}{(q_1 - q_0)(p_1 - p_0)} \ln (q_1/q_0).$$

(ii) If $\ln q = a + bp$, then $E(p) = bp$.

$$E_L = \frac{1}{p_1 - p_0} \cdot b \int_{p_0}^{p_1} p\, dp = \frac{(q_1 - q_0)(p_1 + p_0)}{2(p_1 - p_0)} .$$

(iii) If $\ln q = a + b \ln p$, then $E(p) = b$, for all p.

Thus $E_L = \frac{b(p_1 - p_0)}{p_1 - p_0} = b$.

Solving for b from $\ln q_1 = a + b \ln p_1$

and $\ln q_0 = a + b \ln p_0$,

we get $E_L = b = \frac{\ln (q_1/q_0)}{\ln (p_1/p_0)}$. ◻

b. Minimum Absolute Deviation Elasticity Estimators (E_M)

The MAD elastimators, E_M, solve the following problem.

$$\min_{E_M} \int_{p_0}^{p_1} |E(p) - E_M| \, dp \qquad (2.8)$$

In general, (2.8) is difficult to solve and thus we restrict attention to monotonic $E(p)$*. This is formalized in Assumption 1.

* All special cases considered have this property. Furthermore, if $q = f(p)$ is twice differentiable monotonicity obtains if the following expression does not change sign:

$$\frac{pq''}{q'} + 1 - E(p)$$

Assumption 1:

$E(p)$ is monotonic in p on $[p_1, p_0]$.

Now if $E(p)$ is monotonic then we can restrict attention to estimators in the interval $[E(p_1), E(p_0)]$, since, clearly, E_M will lie within this interval. Further, given the continuity of $E(p)$, there exists $\bar{p}(E_M)$ for $E_M \in [E(p\ ), E(p\ )]$ such that $E(\bar{p}) = E_M$.

Thus we can rewrite (2.8) as follows.

$$\min_{E_M} \int_{p_0}^{\bar{p}} (E(p)-E_M)dp + \int_{\bar{p}}^{p_1} (E_M - E(p))dp \quad (2.9)$$

for $E(p)$ monotonically decreasing and

$$\min_{E_M} \int_{p_0}^{\bar{p}} (E_M - E(p))dp + \int_{\bar{p}}^{p_1} (E(p) - E_M)dp \quad (2.10)$$

for $E(p)$ monotically increasing.*

We now state the main result of this section and prove it.

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* Since $E_M = E(p)$ when the latter is constant, to minimize (2.8) we need not discuss the constant case.

Theorem 3

Under assumption 1, $E_M = E(\hat{p})$ for $\hat{p} = \frac{p_1 + p_0}{2}$.

Proof

We prove the theorem for monotonically decreasing $E(p)$ only, since the proof in both cases is similar.

Let $\hat{E}_M = E(\hat{p})$.

Now suppose that (2.9) is minimized at $E_M^* = E(p^*)$.

Case I, $p^* > \hat{p}$

The expression in square brackets in (2.9) may be written as

$$\int_{p_0}^{p_1} E(p)dp - 2\int_{\hat{p}}^{p_1} E(p)dp + E(\hat{p})(p_1+p_0-2\hat{p}) \text{ for } \bar{p} = \hat{p} \text{ and}$$

$$\int_{p_0}^{p_1} E(p)dp - 2\int_{p}^{p_1} E(p)dp + 2\int_{p}^{p^*} E(p)dp + E(p^*)(p_1+p_0-2p^*)$$

for $\bar{p} = p^*$. Substracting the latter from the former we get

$$-2\int_{\hat{p}}^{p^*} E(p)dp + E(\hat{p})(p_1+p_2-2\hat{p}) - E(p^*)(p_1+p_0-2p^*) \qquad (2.11)$$

which is positive if the latter is a minimum.

Now, since $E(p)$ is monotonically decreasing,

$$-2\int_{\hat{p}}^{p^*} E(p)dp < -2\int_{\hat{p}}^{p^*} E(p^*)dp = 2E(p^*)(p^*-\hat{p}).$$

Thus, $-2E(p^*)(p^*-\hat{p}) - E(p^*)(p_1+p_0-2p^*) > 0$

(The second term drops out since $\frac{p_1+p_0}{2} = \hat{p}$).

That is $-E(p^*)(p_1+p_0-2\hat{p}) > 0$

Since the left hand side is zero this is a contradiction.

Thus $p^* \leq p$.

Case II. $p^* < p$.

Following a similar procedure as before we conclude

$$2\int_{p^*}^{\hat{p}} E(p)dp - E(p^*)(p_1+p_0-2p^*) > 0 ,$$

or $2E(p^*)(\hat{p}-p^*) - E(p^*)(p_1+p_0-2p^*) > 0$.

which is again a contradiction.

Thus (2.9) is minimized at $E(\hat{p}) = E_M$.

Once again we look at the special cases of (2.5) to (2.7). The results are summarised in the next theorem.

<u>Theorem 4</u>

(i) In the linear case $E_M = E_A$. (See(1.3)).

(ii) In the semi-log case $E_M = E_L$

(iii) In the double-log case $E_M = \ln(q_1/q_0)/\ln(p_1/p_0)$

Proof: Trivial. $\square$

Minimax Elasticity Estimators (E_X).

We now restrict attention, once again, to monotonic (but not necessarily continuous) functions.

Now, if $E(p)$ is monotonic, it is clear, first of all, that $[E(p_1), E(p_0)]$ will contain E_X. Secondly, it is also clear that the maximum deviation will occur at $E(p_1)$ or $E(p_0)$ when E_X is an interior point. Thus the minimax estimator will satisfy:

$$E(p_0) - E_X) = (E_X - E(p_1) \tag{2.12}$$

Thus we have

Theorem 5 $E_X = \frac{E(p_1) + E(p_0)}{2}$ if Assumption 1 is satisfied.

The special cases fall out immediately.

Theorem 6

(i) In the linear case $E_X = \frac{(q_1-q_0)(p_1q_0 + p_0q_1)}{2\ q_1q_0(p_1-p_0)}$

(ii) In the semi-log case $E_X = E_L$

(iii) In the double-log case $E_X = \frac{\ln(q_1/q_0)}{\ln(p_1/p_0)}$.

Proof Upon direct substitution, the results follow. □

d. Other Elasticity Estimators and Relationships Between Elasticity Estimators

By application of Jensen's inequality for concave and convex functions, the following results is immediate.

Theorem 7 Under Assumption (1)

(i) $E_M = E_L = E_X$ for linear $E(p)$ on $[p_0, p_1]$.

(ii) $E_M > E_L > E_X$ for strictly concave $E(p)$ on $[p_0, p_1]$

and (iii) $E_M < E_L < E_X$ for strictly convex $E(p)$ on $[p_0, p_1]$

Since the least squares estimator is found to be the average elasticity, we may as well formalize this.

Theorem 8

The least squares estimator is the Average Elasticity on $[p_0, p_1]$. (See Theorem 1).

Also, the constant elasticity estimator, that is the elasticity estimator when the elasticity is assumed to be constant (and $q=f(p)$ has the double log form) has been found previously.

Theorem 9 If $E(p)$ is assumed constant on $[p_0, p_1]$, then $E(p) = \ln(q_1/q_0) / \ln(p_1/p_0)$.

Proof By direct substitution. □

Finally, from Theorem 4, $E_A = E_M$ in the linear case. The extremely special nature of the arc elasticity estimator is thus revealed.

We now address an additional justification for the average elasticity estimator which provides further evidence of its possible desirability.

On the interval $[p_0, p_1]$ we may write $E(p) = \bar{E} + Up$, where Up is the deviation of E(p) from its average value $\bar{E}$ (or E_L). In the absence of any knowledge of the nature of the E(p) function, Up may be thought of as a random variable with mean zero. This is, of course, a random walk model. The sample average, as is well known, is the best unbiased estimator of $\bar{E}$ under a wide variety of distributions of Up. This argument provides a powerful additional justification for the use of the average elasticity concept.

e. A 'Goodness of fit' statistic

We use the statistic described below to derive 'goodness of fit' results for the three special functions discussed above. The statistic is motivated as follows. In any practical application, the true functional form is not known. Thus, the elasticity estimator used should, as far as possible, give results that are 'close' to the true elasticity values on the interval defined by the observations, regardless of the true functional form. To measure 'closeness', the R^2 statistic is a natural choice. The analog of the R^2 statistic, in our case, is given by the following formula.

$$G_X = 1 - \frac{\int_{p_0}^{p_1} (E(p) - X)^2 \, dp}{\int_{p_0}^{p_1} (E(p))^2 \, dp} \qquad (2.13)$$

where $E(p)$ is the elasticity corresponding to ~~one~~ of the special cases (2.5) - (2.7) and X is any elasticity estimator. We may then use the average value of G over the 3 functional forms to choose an estimator.

There are 5 elasticity estimators and three functions. The 15 formulae corresponding to these are presented in <u>Table 1</u>*. Instead of analytical comparisons of the Gs, we provide numerical results.

Note that G_X can, in fact, be negative. To see this, rewrite G_X as

$$X\,(2 \int E(p)dp - X(p_1 - p_0)) \,/ \int (E(p))^2 \, dp.$$

Thus if $X < 0$ $G_X \gtrless 0$ as $\frac{2}{p_1 - p_0} \int E(p)dp \lessgtr X$,

and if $X > 0$ $G_X \gtrless 0$ as $\frac{2}{p_1 - p_0} \int E(p)dp \gtrless X$.

That is, if X is absolutely greater than twice the absolute average value of $E(p)$, then G_X will be negative. Obviously, in this case, X will not be a very good estimator of $E(p)$.

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*The least squares estimator has the lowest G-statistic in at least one case. However, the <u>average G</u> need not be lowest across functional forms.

E_L	E_M	E_X	E_{SL}	E_{DL}
Linear function				
$\frac{E_L^2}{D}$	$\frac{E_M(2E_L-E_M)}{D}$	$\frac{E_X(2E_L-E_X)}{D}$	$\frac{E_{SL}(2E_L-E_{SL})}{D}$	$\frac{E_{DL}(2E_L-E_{DL})}{D}$
Semi-log function				
$E_L(2E_{SL}-E_L)F$	$E_M(2E_{SL}-E_M)F$	$E_X(2E_{SL}-E_X)F$	$\frac{3(p_1+p_0)^2}{4(p_1^2+p_1p_0+p_0^2)}$	$\frac{3\left((p_1+p_0)\ln(p_1/p_0)-(p_1-p_0)\right)}{\ln^2(p_1/p_0)(p_1^2+p_1p_0+p_0^2)}$
Double log function				
$\frac{E_L(2E_{DL}-E_L)}{E_{DL}^2}$	$\frac{E_M(2E_{DL}-E_M)}{E_{DL}^2}$	$\frac{E_X(2E_{DL}-E_X)}{E_{DL}^2}$	$\frac{4(p_1-p_0)G-G^2}{4(p_1-p_0)^2}$	1.0

<u>Notes:</u>

$D = 2E_L + a^2/q_1q_0 - 1$

$F = 3(p_1+p_0)^2 / 4(p_1^2+p_1p_0-p_0^2)$

$G = \ln(p_1/p_0$

3. Numerical Comparisons

Below we present some numerical comparisons of the performance of the five elasticity formulae in theorems 2, 4 and 6. We consider 6 cases each of linear, semilog and double log functions where the 6 cases have been so chosen that elasticities are

(a) positive and elastic, unitary elastic and inelastic (approximately in the unitary case).

and (b) negative and elastic, unitary elastic and inelastic (again, approximately in the unitary case).

Furthermore, we confine our attention to positive values of p and q only. The data set used along with the functions used to generate the data are presented in Table 2. In Table 3 the computed elasticity values* are presented. Finally, in Table 4, G_X statistics are presented.

It is of interest to note that the linear minimax estimator performs the most poorly for all three functions. Further, in two cases with the semi-log function, the G_X statistic is negative in Table 4.

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* A pocket calculator was used.

Table 2: Data Set for Computations

Serial Number	p_1	p_0	q_1	q_0	a	b
Linear function ($q = a+bp$)						
1	6	5	1.2	0.2	-4.8	1
2	6	5	5.8	4.8	0.2	1
3	6	5	66	65	60	1
4	6	5	0.5	1.5	6.5	-1
5	6	5	6	7	12	-1
6	6	5	59	60	65	-1
Semi-log function ($\ln q = a+bp$)						
7	6	5	1100	3	-28.4237	5.9044
8	6	5	40	20	-0.47	0.6931
9	6	5	4	3.8	1.0785	0.0513
10	6	5	18	1100	27.5665	-4.1127
11	6	5	12	16	4.2110	-0.2877
12	6	5	3.8	4	1.6428	-0.0513
Double log function ($\ln q = a+b \ln p$)						
13	6	5	20	8	-6.0091	5.0257
14	6	5	18.5	15.5	1.179	0.9703
15	6	5	9	8.5	1.6355	0.3137
16	6	5	8	20	11.0843	-5.0257
17	6	5	15	18	4.4998	-0.9999
18	6	5	8.5	9	2.7018	-0.3137

Table 3: Computed Elasticity Values

Serial Number	E_L	E_M	E_X	E_{SL}	E_{DL}
1	9.6004	7.8571	19.0000	9.8547	9.8277
2	1.0378	1.0377	1.0381	1.0401	1.0377
3	0.0840	0.0840	0.0840	0.0840	0.0840
4	-6.1410	-5.5000	-7.6666	-6.1346	-6.0256
5	-0.8498	-0.8462	-0.8571	-0.8608	-0.8458
6	-0.0925	-0.0924	-0.0925	-0.0924	-0.0921
7	30.5063	10.9402	917.1585	32.4745	32.3851
8	1.7726	3.6666	4.7500	3.8123	3.8015
9	0.2819	0.2821	0.3000	0.2821	0.2814
10	-23.744	-10.6458	-182.7924	-22.6196	-22.5574
11	-1.5891	-1.5714	1.6250	-1.5823	-1.5780
12	-0.2823	-0.2821	0.2829	-0.2821	-0.2814
13	4.9706	4.7143	5.5500	5.0396	5.0257
14	1.0295	0.9706	0.9709	0.9731	0.9703
15	0.3141	0.3143	0.2876	0.3144	0.3137
16	-5.1086	-4.7143	-6.0000	-5.0396	-5.0257
17	-1.0055	-0.9999	-1.0167	-1.0028	-0.9999
18	-0.3146	-0.3143	-0.3574	-0.3144	-0.3137

Notes: (1) E_{SL} : Common Elasticity formula for semi-log function

(2) E_{DL} : Common Elasticity formula for double log function

(3) The serial numbers correspond to table 2.

Table 4: Computed Values of G_X.

Serial Number	E_L	E_M	E_X	E_{SL}	E_{DL}
Linear Case					
1	0.8071	0.7805	0.5518	0.3065	0.8058
2	1.0000	1.0000	1.0000	1.0000	1.0000
3	0.9722	0.9722	0.9722	0.9722	0.9722
4	0.8760	0.8664	0.8219	0.8760	0.8757
5	0.9901	0.9900	0.9900	0.9899	0.9690
6	1.0000	0.9884	1.0000	0.9884	0.9767
AVERAGE VALUE	0.9409	0.9329	0.8893	0.9388	0.9332
Semi-log Case					
7	0.9936	0.5588	-739.148	0.9973	0.9972
8	0.7118	0.9959	0.9370	0.9973	0.9972
9	0.9973	0.9973	0.9935	0.9973	0.9972
10	0.9948	0.7178	-49.01	0.9973	0.9972
11	0.9973	0.9973	0.9966	0.9973	0.9972
12	0.9973	0.9973	0.9973	0.9973	0.9972
AVERAGE VALUE	0.9487	0.8774	negative	0.9973	0.9972
Double log case					
13	0.9999	0.9962	0.9891	0.9999	1.0000
14	0.9963	1.0000	1.0000	0.9999	1.0000
15	1.0000	1.0000	0.9932	0.9999	1.0000
16	0.9997	0.9962	0.9624	0.9999	1.0000
17	1.0000	1.0000	0.9997	0.9999	1.0000
18	1.0000	1.0000	0.9807	0.9999	1.0000
AVERAGE VALUE	0.9993	0.9987	0.9875	0.9999	1.0000

In Table 4, the average G_X values for each of the three functions and for each estimator are presented. If we compute the grand mean for all three cases, E_{SL} is seen to perform best followed by E_{DL} and then E_L. E_X is always the worst. Alternatively, it may be seen that across functional forms, if we rank the average values and compute their average, the same ranking of estimators is found as above. It is significant that the arc elasticity has the second lowest rank in all three cases.

On the basis of these computations, therefore, it appears that the formula E_{SL} is the most robust, where

$$E_{SL} = \frac{(p_1+p_0)\ \ln(q_1/q_0)}{2(p_1-p_0)} \ .$$

Conclusions

In this paper, 5 alternative estimators of elasticities are derived from widely accepted statistical criteria. Some numerical comparisons of these elasticity estimators are carried out and the robustness of the estimators with respect to misspecification is analysed. Subject to further work, it is found that the estimator E_{SL} in (3.1) is the most robust. The arc elasticity performs extremely poorly. These results are being extended to elasticity estimators in the standard regression model in on-going work by the authors.