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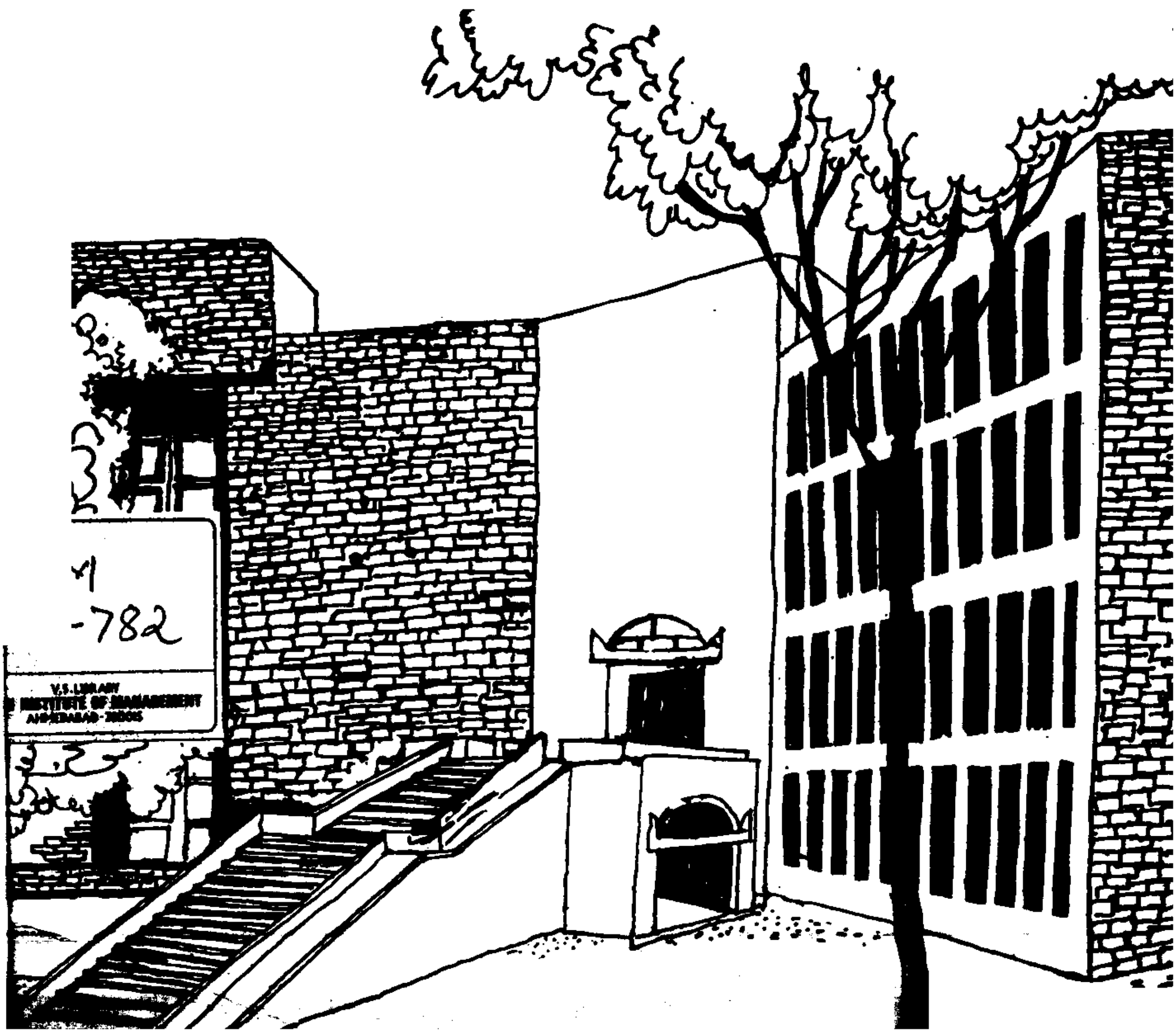

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FAIR ALLOCATIONS AND DISTORTION OF UTILITIES: A NOTE

By

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ABSTRACT

Given two agents with von Neumann-Morgenstern utilities who wish to divide n commodities, consider the two-person non-cooperative game with strategies consisting of concave, increasing von Neumann-Morgenstern utility functions and whose outcomes are fair allocations to the commodity division problem determined by the strategies used. It is shown that any equal income competitive equilibrium allocation for the true utilities is a Nash equilibrium outcome for the non-cooperative game.

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Introduction : Consider the problem of dividing a fixed amount of goods among two agents. Fundamental work in this area done by Foley (1967), Varian (1974), Pazner and Schmeidler (1968) discusses how to divide the total available product fairly between the two agents.

It is often the case in economic or game theoretical models that predictions are based on information that is not observable. For example, Nash's (1950) theory of bargaining determines an outcome that depends on the von-Neumann-Morgenstern utility functions. Kurz (1977, 1980), Crawford and Varian (1979), Sobel (1981) analyse outcomes of the bargaining solutions over the division of commodities between two agents. Their method like our's in this paper is to embed the original game into a non-cooperative distortion game in which the players' strategies consist of utility functions that may be distorted from their true utilities for strategic purposes. The outcomes are given by the solution to the underlying game determined by the reported utilities. If the Nash equilibria of the distortion game share common properties, then a description of the original game situation has been made without relying on information about the unobserved utility functions.

The problem considered in this paper may be viewed as an arbitration problem under ignorance. An arbitrator is assigned the task of determining a fair outcome to a division of commodities problem. A possible technique for the arbitrator who knows what the total supply of the commodities is, would be to ask the players to report their true utility functions and then determine a fair outcome to the resulting bargaining game. In such a situation the agents will be playing the distortion game described here.

Here we assume t at agents have perfect information about the game situation they are facing.

Closely related to our results are those of Thomson (1979a and b). Thomson studies the Nash equilibria for the distortion game derived from a class of performance correspondences that yeild individually rational and Pareto-efficient outcomes. Thomson (1979a) finds that if the reported utility functions are restricted to be twice continuously differentiable, concave and have the transferable utility (t.u) property, then the Nash equilibria for the distortion game derived from the Shapley value with fixed initial endowments are exactly th constrained competitive allocations with respect to those endowments. This result is generalized to a broader class of performance correspondences in Thomson (1979b).

In this paper we show that for the class of strategies considered here any equal - income competitive equilibrium allocation for the true utilities in a Nash equilibrium outcome for the non-cooperative game. Throughout we assume that the supply of goods is fixed and known to everybody.

In Hurwicz (1972), it is shown that for the same class of envorinments, the individually rational and Pareto optimal correspondence (in the context of a private ownership economy) is non-implementable. The same example (example D) in his paper would show that the Pareto correspondence for the class of economies considered by us is non-implementable. What we have shown however is that by suitably restricting the performance correspondence, we are able to implement our desired social goals. The reason behind this is simple. The Pareto correspondence being too large, any player given the Nash strategy of the other player can deviate to a strategy which yields a better non-Pareto optimal allocation with regard to true preferences and yet remains Pareto-optimal with regard to stated preferences. By reducing the size of the performance correspondence, we are not eliminating better allocations for the players. We only guarantee that these better allocations do not belong to the image of the performance correspondence obtained by an unilateral derivation from the Nash strategies.

This resolves a very important question with regard to implementing fair allocations in exchange economies.

2. <u>Definitions and Notation</u> : Consider two agents with von Neumann-Morgenstern utility functions who are to divide a bundle of n-commodities. Units are chosen so that there is exactly one unit of each commodity. Letting $\underline{a} = (a,\ldots,a)$, an outcome will be an element of the set

$$T = \left\{ x \in \mathbb{R}^n / \underline{0} \leq x \leq \underline{1} \right\}$$

where agent 1 receives x and agent 2 receives $\underline{1}-x$. We assume that the true utility function of player 1 is denoted u, and the true utility function of player 2 is denoted v. These functions are assumed to be concave and strictly increasing in T.

The players report utilities that are restricted to lie in the class $\mathcal{U}$, where $\mathcal{U}$ consists of those functions $U: T \longrightarrow [0,1]$ such that

a) U is continuous, monotonic (i.e. $x \leq x'$ and $x \neq x'$ implies $U(x) < U(x')$)
b) U is normalized so that $U(\underline{0}) = 0$ and $U(\underline{1}) = 1$.

The distortion game for the fair division problem is played by each agent revealing a utility function in $\mathcal{U}$. Typically, U will denote the function revealed by player 1; V that of player 2. Given these reports, a set of outcomes $F(U,V)$ is selected.

An outcome $x \in T$ is efficient if there is no other outcome $y \in T$ with $U(y) \geq U(x)$ and $V(\underline{1} - y) \geq V(1-x)$ one of them being a strict inequality.

An outcome $x \in T$ is equitable if $U(x) \geq U(\underline{1}-x)$ and $V(\underline{1}-x) \geq V(x)$.

If an outcome $x \in T$ is both equitable and efficient, we will say x is <u>fair</u>.

Let

$$F(U,V) = \{ x \in T / x \text{ is fair} \}.$$

<u>Definition</u> : The strategies (U^*,V^*) constitute a <u>Nash equilibrium for the distortion game</u> determined by F if and only if

(a) $(U^*,V^*) \in \mathcal{U} \times \mathcal{U}$

(b) there exists an $x^* \in F(U^*,V^*)$ such that x^* solves $\max u(x)$

subject to x $F(U,V^*)$

and $\underline{1} - x^*$ solves $\max v(y)$

subject to $\underline{1} - y \in \bigcup_{U \in \mathcal{U}} F(U^*,V)$.

3. <u>Main Results</u> : We begin with a definition of equal-income competitive equilibrium (EICE).

<u>Definition</u> : An <u>equal-income competitive equilibrium</u> (EICE) is a pair, (p^*,x^*) where (a) $p^* \in IR^n$, $p^*_i \geq 0$, $\sum_{i=1}^{m} p^*_i = 1$;

(b) $x^* \in T$;

(c) x^* solves:

$\max u(x)$ subject to $p^* . x \leq \frac{1}{2} p^* . \underline{1}$ and $x \in T$

(d) $1 - x^*$ solves:

$\max v(y)$ subject to $p^* . y \leq \frac{1}{2} p^* . \underline{1}$ and $y \in T$

The vector x^* will be referred to as the competitive allocation.

On occasion, a vector $p = (p_1, \ldots\ldots, p_n)$ will be used to refer to the linear function from T to IR, where

$$p(x) \equiv p.x \equiv \sum_{i=1}^{n} p_i x_i .$$

Theorem 1 : Suppose that the true preferences are monotonic and (p^*, x^*) is an EICE for true preferences u and v. Then x^* is fair.

Proof: Varian (1974): First we will show that x^* is strongly efficient. Assume not ; then there is some allocation y such that $u(y) \geq u(x^*)$, $v(\underline{1} - y) \geq v(\underline{1} - x^*)$, one of them being a strict inequality. We can choose y so that it itself is efficient.

Suppose without loss of generality $u(y) > u(x^*)$.

$$\therefore\ p^*.y > p^*.x^* = \tfrac{1}{2}\, p^*.\underline{1}$$

by the definition of EICE and monotonicity.

If $v(\underline{1} - y) > v(\underline{1} - x^*)$ then

$$p^*.(\underline{1} - y) > p^*.(\underline{1} - x^*) = \tfrac{1}{2}\, p^*.\underline{1}$$

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for similar reasons,

$$\therefore\ p^*.\underline{1} = p^*.y + p^*.(\underline{1} - y) > p^*.x^* + p^*.(\underline{1} - x^*) = p^*.\underline{1}$$

a contradiction.

So suppose $v(\underline{1} - y) = v(\underline{1} - x^*)$.

If $p^*.\ (\underline{1}-y) < p^*(\underline{1}-x^*)$, agent 2 could afford to buy a slightly more expensive bundle, and by monotonicity he could find a bundle strictly better than $\underline{1}-x^*$, contradicting that x^* is a competitive allocation.

Thus,

$$p^*.(\underline{1}-y) \geq p^*(\underline{1}-x^*).$$

Once again,

$$p^*.\underline{1} = p^*.y + p^*.(\underline{1}-y) > p^*.x^* + p^*.(\underline{1}-x^*) = p^*.\underline{1}$$

a contradiction.

Hence x^* is efficient.

To show that x^* is also equitable, we suppose that agent 1 envies agent 2.

$\therefore\ u(\underline{1}-x^*) > u(x^*)$ as so by the definition of EICE,

$$p^*.(\underline{1}-x^*) > p^*.x^* = p^*.(\underline{1}-x^*),$$

which is a contradiction.

The main theorem can now be stated

<u>Theorem 2</u>: If $(p^*; x^*)$ is an EICE, then (p^*,p^*) is a Nash equilibrium for the distortion game.

The proof of the above theorem uses the following lemmas.

<u>Lemma 1</u> : If $x \in F(U,p^*)$ then $p^*.(\underline{1}-x) \geq \frac{1}{2}$.

If x $F(p^*,V)$ then $p^*.x \geq \frac{1}{2}$.

Proof: Suppose $x \in F(U,p^*)$ and $p^*.(\underline{1}-x) < \frac{1}{2}$.

Since $p^*. \underline{1} = 1.$, we get $p^*.x > \frac{1}{2}$

Hence player 2 envies player 1 and so $x \notin F(U,p^*)$, a contradiction

$\therefore\ p^*.(\underline{1}-x) \geq \frac{1}{2}$

Similarly $x \in F(p^*, V)$ implies $p^*.x \geq \frac{1}{2}$.

Lemma 2 : $F(p^*,p^*) = \left\{ x \in T: p^*.x = \frac{1}{2} \right\}$

Proof: By lemma 1 and symmetry,

$F(p^*,p^*) \subseteq \left\{ x \in T: p^*.x = \frac{1}{2} \right\}$.

Now, let $p^*.x = \frac{1}{2}$.

$p^*.(\underline{1} - x) = \frac{1}{2}$.

Hence x is equitable.

Let $y \in T$ with $p^*.y > p^*.x.$, $p^*.(\underline{1}-y) \geq p^*.x$

$\therefore\ 1 = p^*.y + p^*.(\underline{1}-y) > p^*.x + (\underline{1}-x) = 1$,

which is a contradiction.

Similarly $p^*.(\underline{1}-y) \geq p^*.(1-x)$, $p^*.y \geq p^*.x$ leads to a contradictionn.

$\therefore$ x is effieient.

$\therefore\ x \in F(p^*,p^*)$.

Hence the lemma.

Proof of theorem 2 :- Let $x \in F(U,p^*)$.

$p^*.(\underline{1}-x) \geq \frac{1}{2}$ implies $p^*.x \leq \frac{1}{2}$.

But x^* is a competitive allocation.

So, x^* solves:

$$\max u(x)$$

$$\text{subject to } p^*.x \leq \tfrac{1}{2}, \quad x \in T.$$

$\therefore$ x^* solves:

$$\max U(x)$$

$$\text{subject to } x \in F(U,p^*) \quad \forall\, U \in \mathcal{U}.$$

Since u and v are monotone and x^* is a competitive allocation,

$$p^*.x^* = p^*.(\underline{1}-x^*) = \tfrac{1}{2}.$$

$\therefore$ $x^* \in F(p^*,p^*)$, by Lemma 2

Similarly we can show that if $y \in F(p^*,V)$, then

$$v(\underline{1}-y) \leq v(\underline{1}-x^*).$$

Hence (p^*,p^*) is a Nash Equilibrium for the distortion game. This establishes the theorem.

Q.E.D.

Theorem 2 has a partial converse.

Theorem 3 :- If (p,q) is a Nash equilibrium for the distortion game, and if p and q satisfy condition (a) of the definition of an EICE, then there exists an $x^* \in F(p,q)$ such that x^* is an EICE allocation for true preferences.

Proof : Let (p,q) be a Nash equilibrium. Then there exists $x^* \in F(p,q)$ such that

$$u(x^*) \geqslant u(x) \quad \forall\ x \in F(U,q) \text{ and for all } U \in \mathcal{U}.$$

By Lemma 1, $q(\underline{1}-x) \geqslant \frac{1}{2}$, $q(\underline{1}-x^*) \geqslant \frac{1}{2}$.

Similarly $x \in F(p,V)$ implies $p.x \geqslant \frac{1}{2}$, $p.x^* \geqslant \frac{1}{2}$

and $v(1-x^*) \geqslant v(1-x)$.

Let $x \in T$ with $q\ (\underline{1}-x) \geqslant \frac{1}{2}$

Since $F(q,q) = \left\{ x \in T : q.x = \frac{1}{2} \right\}$,

$u(x^*) \geqslant u(x)$ if $x \in T$ and $q.(\underline{1}-x) = q.x = \frac{1}{2}$.

But u is an increasing function.

$\therefore$ x^* solves:

$$\max u(x)$$
$$\text{s.t. } q.x \leqslant \tfrac{1}{2}$$
$$x \in T$$

Similarly $\underline{1} - x^*$ solves:

$$\max v(y)$$
$$\text{s.t. } p.y \leqslant \tfrac{1}{2}$$
$$y \in T$$

Further $q.x^* = p.x^* = \frac{1}{2}$

To prove the theorem, it suffices to show that $p = q$. Now x^* must be efficient with respect to the utility functions p and q. Also equal division is efficient with respect to the utility functions p and q.

(Suppose not. Then there exists $y \in T$ with $p.y \geq \frac{1}{2}$, $q.(\underline{1}-y) \geq \frac{1}{2}$ with at least one being a strict inequality. But this would contradict the efficiency of x^* with respect to p and q).

Since equal division is efficient, and since p and q are normalized, p = q.

Q.E.D.

4. Conclusion :

In this paper we resolve a very important question with regard to the implementation of fair (in the sense of Varian (1974)) allocations in an exchange economy. Consider an economy where a fair allocation of a certain fixed supply of goods has to be implemented by an arbitrator. To be able to abhieve a fair allocation, the arbitrator needs to know the true utility functions of the agents. If the arbitrator declares that he will implement the equal income competitive equilibrium on the basis of reported utilities the agents will report linear utility functions. The EICE with these reported preferences as a basis, gives rise to fair allocations for true preferences.

Finally, it is worth mentioning that in spite of the marked similarity in our approach with that of Sobel (1981) and a fair amount of similarity in our results with those of his, (especially Theorem 3 of Sobel (1981)), the underlying games are completely different. Sobel proves that for a certain class of distortion games, the Nash equilibrium outcomes are envy-free with respect to the true preferences of the agents. His admissible class of distortion games does not include ours. Thus in some senses our result may be considered to be an extension of earlier work done in this area. The generalization of the above results to the situation with more than two agents is analogous and modulo appropriate changes in definitions (see Varian (1974) for example) is quite immediate.

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