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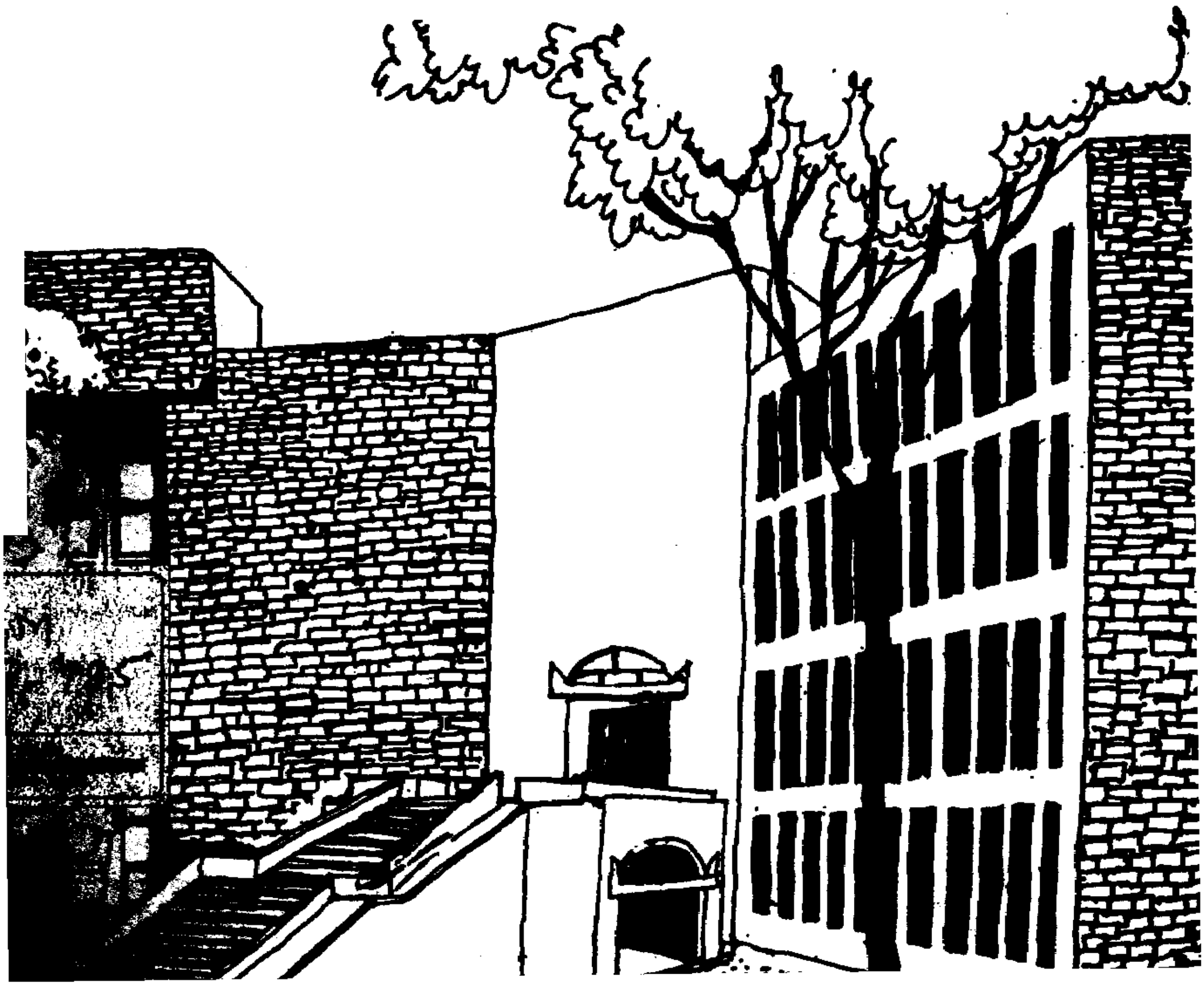

PROCEDURES FOR GENERATING AN INFORMATIONALLY EFFICIENT EQUITABLE SOLUTION

By

Somdeb Lahiri

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INDIAN INSTITUTE OF MANAGEMENT
AHMEDABAD-380 056
INDIA

ABSTRACT

In this paper we consider two games which generate A-envy free allocations in a pure exchange economy with a fixed supply of resources and agents' preferences being representable by utility functions. The first game is "classical divide and choose" whereas the second game is "equal division divide and choose". A detailed analysis and comparison of the relative merits of the two games follow.

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Introduction :

Methods for generating solutions to equity problems have been advanced in the case of fair allocations by Crawford ((1977), (1980)) and in the case of Pareto efficient egalitarian equivalent allocations by Crawford (1979). Prior to that a series of works (Dubins and Spanier [1961], Kolm [1972], Kuhn [1967], Luce and Raiffa [1957], Rawls [1971], Singer [1962], Steinhaus [1948]) have pondered on the problem of fair division of a fixed supply among a group of agents. The methods were elegant. However, the informational requirement for implementing the equity criteria usually turned out to be exorbitant. Pazner (1977) proposed the concept of per-capita fairness which was informationally much less demanding. This was followed by an informationally efficient equity criterion by Thomson (1982), which for large economies was shown to approximate the per-capita fairness criteria of Pazner. In this paper, we shall concentrate on non-cooperatively implementing this final solution to the equity problem.

The "divide-and-choose" method which has played an important role in the literature on fair division seems an impartial technique for allocating bundles of goods, requires little cooperation from agents, and is nearly free of administrative costs. The method seems to have been underutilized and in the sequel we propose an application of the above mebhod.

If an agent prefers the average bundle of goods of all other agents (except for himself) to his own bundle he will be said to A-envy the other agents. Here 'A' stands for the fact that each agent is using a summary statistic of the consumption of all other agents (i.e. their average consumption) which makes the comparison anonymous. It is also in this sense that this comparison requires less information than finding out whether each agent prefers his own consumption bundle to that of any other. In case an agent does prefer the consumption bundle of any other agent to his own, he is said to envy that other agent. The latter criteria requires n(n-1) comparisons, where as the former requires only n comparions, given that there are n agents in the economy. But, as with all such informationally desirable criteria, the problem is with regard to its implementation.

We assume as always that players seek to obtain the most desirable bundle possible. They are also assumed to behave non-cooperatively, since negotiating a mutually acceptable settlement would be relatively easy if they were willing to cooperate and the method would then be superfluous. We assume that there is a divider (chosen possibly by the toss of a coin) and the rest are choosers. In Section 2 of this paper the divider's problem is formulated and his optimal non-cooperative strategy is characterized. If the divider knows the preferences of the others with certainty, under very general conditions - roughly, that players' behaviour can be described by the maximization of continuous and strongly monotonic utility functions and that goods are homogeneous and perfectly divisible - his optimal strategy involves dividing the bundle so that all the other players are indifferent about their choices. In Section 3 a new game is suggested which remedies some of the undesirable features of classical divide and choose method. This method called equal division divide and choose method, provides individually rational (from equal division), Pareto efficient and A-envy free allocation. Section 4 is the Conclusion.

2. <u>The Pure Trade Divide and Choose Game</u> : Assume that n agents have agreed to share a fixed bundle of homogeneous and perfectly divisible good by a modified divide and choose method, and that their roles have already been determined in some way, perhaps by the toss of an unbiased coin. Each persone behaves non-cooperatively, seeks only to obtain the most desirable bundle possible and has preferences that are representable by a continuous and strongly monotonic (though not necessarily quasi-concave) utility function. In addition, the divider, who without loss of generality, is assumed to be agent 1, is supposed to know the preferences of all other agents with certainty. This is not a completely natural assumption to make in most situations. But studying the certainty

case provides a necessary preliminary to a more general analysis and may serve as a reasonable description of some situations. A final assumption is included to simplify exposition. Whenever a chooser is indifferent between two bundles offered to him he can be counted on to choose the one that the divider would prefer him to. Since it has already been assumed that the chooser's preferences are known with certainty, this assumption is innocuous – a tiny adjustment in the division of agent 1 could induce the choosers to make the desired choice without perceptibly altering any player's consumption or welfare. This assumption allows us to deal with maxima instead of suprema, and greatly simplifies the statements of some of the results.

In the sequel, the following vector notation is used: if

$$\underline{a} \equiv (a_1, \ldots, a_r) \text{ and } \underline{b} \equiv (b_1, \ldots, b_r),$$

$\underline{a} \leqq \underline{b}$ means $a_i \leqq b_i$, $(i = 1, \ldots, r)$, $\underline{a} \cdot \underline{b} \equiv \sum_{i=1}^{r} a_i b_i$, and $\underline{a}\,\underline{b} \equiv (a_1 b_1, \ldots, a_r b_r)$. $\underline{0}$ and $\underline{1}$ denote a vector of zeros and a vector of ones, whose dimensionalities should be inferred from the context.

Units are chosen so that the vector of goods to be allocated is $\underline{1}$. A division by the divider (agent 1) will be represented by an n-tuple of l-vectors $(z_1, \ldots, z_n)$, where $z_i \in IR^l_+$ the consumption space of agent i. Let $U_i : IR^l_+ \to IR$ be a continuous utility function for agent i. We assume that U_i is monotone i.e. $z, z' \in IR^1_+$, $z \geqq z'$ $z \neq z'$, $U_i(z) \geq U_i(z')$. Then agent i's optimal non-cooperative strategy in the game is a solution of the following programming problem:

$$\max \; U_1(z_1)$$
$$0 \leqq z_1 \leqq 1$$

$$\text{subject to } U_i(z_i) \geqq U_i\left(\frac{1 - z_i}{n-1}\right), \quad i > 1 \qquad \text{(A)}$$

$$\sum_{i=1}^{n} z_i = 1 \, .$$

A solution to (A) always exists, since a continuous function defined on a compact, nonempty set always takes on a maximum value at some point in the set. Proposition 2 of Thomson (1982) shows that the constraint set is nonempty. The following theorem provides an optimality condition that any solution must satisfy.

<u>Theorem 1</u> :- Any solution $(z^*_1, \ldots, z^*_n) \in IR^{1n}_+$ to (A) must satisfy

$$U_i(z^*_i) = U_i\left(\frac{1 - z^*_i}{n-1}\right) \quad \ldots. \qquad (1)$$

for $i > 1$.

<u>Proof</u> : A straightforward application of continuity and monotonicty.

If U^i is differentiable for $i > 1$, total differentiation of (1) reveals that the locus of points in IR^1_+ which satisfy (1) need not be a straight line, and has the same slope as the indifference curve of agent i at the point $\frac{1}{n}\underline{1}$ through which it passes. In addition, since U^i is quasi-concave.

$$U_i\left(\frac{1}{n}\underline{1}\right) \geq U_i\left(\frac{1}{n}z' + \frac{n-1}{n}\left(\frac{\underline{1}-z'}{n-1}\right)\right) \geq U_i(z') = U_i\left(\frac{\underline{1}-z'}{n-1}\right)$$

where z' is any point which satisfies (1). Thus $U^i(.)$ must be tangent to the locus of all points satisfying (1) at the point $\frac{1}{n}\underline{1}$.

Lemma 1 : By dividing appropriately, agent 1 can enforce any allocation $(z_1, \ldots, z_n) \in IR^{1n}_+$ which satisfies

$$U_i(z_i) \geq U_i\left(\frac{\underline{1} - z_i}{n-1}\right), \quad i = 1, \ldots, n$$

$$\sum_{i=1}^{n} z_i = \underline{1},$$

as the outcome of the above game, where each agent $i > 1$ has a choice between z_i and $\frac{\underline{1} - z_i}{n-1}$.

Proof : Since all such allocations are feasible the proof follows.

It is appropriate at this point to describe the game that is being played. The divider offers each agent i a choice between $z_i \in IR^1_+$

and $\frac{1-z_i}{n-1}$, the latter being the average of what is consumed by all other agents. In order to guarantee that the divider gets his optimal share, he must ensure that each agent i non-cooperatively opts for z_i. This happens when $U_i(z_i) \geq U_i\left(\frac{1-z_i}{n-1}\right)$.

<u>Definition</u> : An allocation $(z_1, \ldots, z_n) \in IR_+^{ln}$ is A-envy free if

(i) $U_i(z_i) \geq U_i\left(\frac{1-z_i}{n-1}\right), \quad i = 1, \ldots, n$

(ii) $\sum_{i=1}^{n} z_i = \underline{1}$.

The next theorem establishes that the allocations generated by the game are A-envy free and that although they need not be Pareto-efficient (as shown by Crawford (1977) for n = 2), they are efficient in a weaker sense. There may exist allocations that both players would prefer to the outcome of the game, but there are no such A-envy free allocations.

<u>Theorem 2</u> : The outcome of the game is an A-envy free allocation, and if agent 1 agrees to break ties for the solution by dividing as the choosers would unanimously prefer, the outcome is an efficient point in the set of all A-envy free allocations.

<u>Proof</u> : Suppose towards a contradiction that $(z^*_1, \ldots, z^*_n)$ is a solution to the problem which is not A-envy free. By Theorem 1, $U_i(z^*_i) = U_i\left(\frac{1 - z^*_i}{n-1}\right)$ for $i > 1$. Hence $U_1\left(\frac{1 - z^*_1}{n-1}\right) > U_1(z^*_1)$.

Define, $y^*_i = \frac{1 - z^*_i}{n-1}$.

Clearly $U_i(y^*_i) = U_i(\frac{1-y^*_i}{n-1})$ for $i > 1$,

$$\sum_{i=1}^{n} y^*_i = 1 \text{ and } U_1(y^*_1) > U_1(z^*_1),$$

contradicting that $(z^*_1, \ldots\ldots, z^*_n)$ is a solution to (A).

Hence $(z^*_1, \ldots\ldots, z^*_n)$ is A-envy free.

To prove the second part, observe that, by Lemma 1, agent 1 can enforce any A-envy free allocation, so there cannot be any A-envy free allocation that agent 1 prefers to the outcome of the game. Thus, either the outcome is efficient in the set of A-envy free allocations or there exists an A-envy free allocation that yields agent 1 the same utility, agents 2 to n at least as much utility as in the given outcome, and at least one agent belonging to the set $\{2, \ldots\ldots, n\}$ strictly more utility. But the latter possibility is ruled out by our assumption about agent 1's tie-breaking, so the outcome is efficient in the set of A-envy free allocations.

The next theorem formalizes the notion that the role of divider is an advantage in the game if preferences are known.

Theorem 3 : In the above game, agent 1 does at least as well in the role of divider as he would in the role of a chooser.

Proof :- Consider the game in which agent 1 is a chooser and one of the agents 2 to n is a divider. The allocation generated by this game is fair by Theorem 2. Therefore by Lemma 1, agent 1 can enforce this allocation when he is a divider, and so must do at least as well as he would in the role of chooser.

Luce and Raiffa's (1957) example [pp.364-365] of the game when the bundle to be divided consists of a single indivisible object shows that some restrictions on goods or preferences are necessary for Theorem 3 to be true. The next theorem compares the allocations generated by this game with those from an equal income competitive equilibrium (EICE). An EICE is A-envy free as shown by Thomson (1982).

<u>Theorem 4</u>:- In the divide and choose game, agent 1 always does at least as well as he would at an EICE.

<u>Proof</u> : Since an EICE is A-envy free it is a feasible allocation for the problem (A). Hence the divide and choose game gives him at least as much utility as he would be getting at an EICE.

3. <u>The Equal Division Divide and Chhose Game</u>:

In spite of its apparant advantages, the classical "divide-and-choose" method is rarely used to resolve bargaining disputes. This could be partly explained because of the inefficiency of the resulting allocations. We shall now modify the divide and choose game and present a different game i.e. the equal division divide and choose game which generates Pareto-efficient allocations.

Once again 1 is the divider, who offers agent $i > 1$ a choice between $\frac{1}{n}\underline{1}$ and z_i. His optimal "division" therefore solves the following problem:

$$\max U^1(z_1)$$

$$\text{s.t. } U_i(z_i) \geq U_i(\tfrac{1}{n}\underline{1}), \quad i = 2,\ldots,n.$$

$$z_i \in IR^l_+, \quad i = 1,\ldots,n \qquad \text{(B)}$$

$$\sum_{i=1}^{n} z_i = \underline{1}.$$

The constraints $U_i(z_i) \geq U_i(\frac{1}{n}\underline{1})$ ensures that each agent $i = 2,\ldots,n$, will voluntarily choose z_i over $\frac{1}{n}\underline{1}$ and thus that agent 1 will get z_1. If agent 1 prefers $\frac{1}{n}\underline{1}$ to all other allocations that are feasible, he can, of course, obtain that bundle by setting $z_i = \frac{1}{n}\underline{1}$, $i=1,\ldots,n$.

__Lemma 2__ : Any solution of (B) must satisfy

$$U_i(z_i) = U_i(\tfrac{1}{n}\underline{1}), \quad i = 2,\ldots,n.$$

__Proof__ : Follows easily from monotonicity and continuity.

Lemma 2 shows that, in solving (B), agent 1 will propose as an alternative to equal division an allocation that yields him the greatest utility while keeping each agent $i = 2,\ldots,n$ on agent i's indifference curve through $\frac{1}{n}\underline{1}$. Therefore, by our assumption that each agent $i = 2,\ldots,n$ chooses as agent 1 prefers when the former are indifferent about their choices, the alternative allocation $(z_1,\ldots,z_n)$ solving (B) is the outcome of the game. We shall now establish some additional properties of the revised game.

Theorem 5 : The solution to EDDC always generates a Pareto-efficient allocation.

Proof : Let $(z^*_1, \ldots, z^*_n)$ be the solution to (B) and suppose that it is not Pareto-efficient. Then there exists an allocation $(z_1, \ldots, z_n)$ such that $\sum_{i} z_i = \underline{1}$ and $U_i(z_i) \geq U_i(z^*_i) \; \forall \; i=1,\ldots,n$

with $U_i(z_i) > U_i(z^*_i)$ for some $i = 1,\ldots,n$.

By monotonicity and continuity we may assume that

$$U_i(z_i) > U_i(z^*_i) \text{ for all } i = 1,\ldots,n \; .$$

But since $(z_1, \ldots, z_n)$ satisfies the constraints of (B), this would contradict that $(z^*_1, \ldots, z^*_n)$ solves (B). Hence $(z^*_1, \ldots z^*_n)$ is Pareto efficient.

Theorem 6 :- EDDC always generates an allocation that is individually rational from ED (equal division).

Proof : Since $z_i = \frac{1}{n}\underline{1}$, $i = 1,\ldots,n$, satisfies the constraints of (B), the theorem is obvious.

Theorem 7 : Let U_i be semi-strictly quasi-concave for $i = 1,\ldots,n$ (i.e. let $z_i, z^1_i \in IR^1_+$ with $U_i(z_i) > U_i(z^1_i)$ and $0 \leq \alpha \leq 1$; then $U_i((1-\alpha) z_i + \alpha z^1_i) > U_i(z^1_i)$). Then EDDC always generates an A-envy free allocation.

Proof: Suppose towards a contradiction that $U_i(\frac{1-z^*_i}{n-1}) > U_i(z^*_i)$ for some $i \in \{1,\ldots\ldots,n\}$ where $(z^*_1,\ldots\ldots,z^*_n)$ solves (B).

Observe that,

$$\frac{1}{n}(\underline{1}) = \frac{1}{n} z^*_i + \frac{n-1}{n} \left(\frac{1-z^*_i}{n-1}\right).$$

$\therefore$ $U_i(\frac{1}{n}\underline{1}) > U_i(z^*_i)$. This contradicts the conclusion of Theorem 6. Hence $(z^*_1,\ldots\ldots,z^*_n)$ is A-envy free as was required to be proved.

The semi-strict quasi concavity of utility function is crucial for Theorem 7. This result is not true otherwise, since Pareto efficiency and A-envy free property may be inconsistent when agents do not have semi-strictly quasi concave utility functions (see Thomson (1982)).

Before continuing with the results of this section, it is convenient to pause for a lemma.

Lemma 3 : By dividing appropriately, agent 1 can achieve any physically feasible allocation that is individually rational from ED (equal division) as the outcome of EDDC.

Proof : Let $(z_1,\ldots\ldots,z_n)$ satisfy $U_i(z_i) \geq U_i(\frac{1}{n}\underline{1})$. Since $(z_1,\ldots,z_n)$ is feasible for (B), the Lemma follows.

The next result shows that EDDC, like classical divide and choose, confers an advantage on the divider.

Theorem 8: In EDDC, the role of divider is an advantage.

Proof : Let $(z_1,\ldots,z_n)$ be the solution to EDDC when agent 1 is a chooser and agent $j(\neq 1)$ is a divider. Clearly, by Theorem 6, $U_i(z_i) \geq U_i(\frac{1}{n}\underline{1})\ \forall\ i$. Hence $(z_1,\ldots,z_n)$ satisfies the constraints of (B), when agent 1 is a divider. Hence agent 1 has an advantage when he is a divider.

It is also clear that unless equal division happens to be Pareto-efficient, there exists $(z_1,\ldots,z_n)$ such that $\sum_{i=1}^{n} z_i = \underline{1}$ and $U_i(z_i) > U_i(\frac{1}{n}\underline{1})$ for all i. As a chooser, agent 1 gets $U_1(\frac{1}{n}\underline{1})$ where as a divider since $(z_1,\ldots,z_n)$ is feasible for (B), he gets at least as much as $U_1(z_1)$. Thus agent 1 does strictly better as a divider than as a chooser.

Theorem 9: Agent 1 does at least as well in classical divide and choose than in EDDC, provided U_i is semi-strictly quasi concave for all i.

Proof: Let $(z^*_1,\ldots,z^*_n)$ be a solution to (B). By Theorem 7, $(z^*_1,\ldots,z^*_n)$ satisfies the constraints of problem (A). Hence the theorem follows.

For the case n = 2, A-envy free allocations are the same as envy free allocation and hence results pertaining to the latter as in Crawford (1977; 1980) are valid here as well. In particular are valid the relations between EDDC and CDC. Before we conclude this section, let us pause for another result.

Theorem 10: EDDC treats agents more unequally than any other allocation that is individually rational from equal division.

Proof : In EDDC, agent 1 can enforece any allocation that is individually rational from equal division (Lemma 3), so he must do at least as well at an EDDC allocation as he would at any other allocation that is individually rational from equal division. Lemma 2 implies that agent i, $i \geq 2$ must do at most as well at an EDDC allocation as he would at the other allocation.

Suppose that EDDC is modified by letting another allocation that is individually rational from equal division (ED) play the role that ED does in EDDC; this allocation may be called the basis allocation of the new device. With obvious changes in wordings the proofs of Lemmas 2 and 3 and Theorems 5,6,7,8,9, 10 are all valid. ED however is a simpler basis for the method than other allocations and seems less arbitrary because of its perfect symmetry. If it is possible to find, and persuade agents to accept another allocation that is individually rational from ED as the basis for an EDDC-type allocation device, this device will have all of EDDC's optimality properties. In fact this new device would treat agents more unequally than EDDC does. The possibility of using allocations other than $\bar{E}D$ as the basis allocation may provide a way of adapting EDDC to situations where agents are not in symmetrical positions. In fact, if the status quo is taken as the basis allocation, EDDC can be viewed as a way of making operational the classical dictum of welfare economics that inefficiencies should be removed as long as those who gain by the change compensate those who lose.

4. Conclusion : In this paper we have analysed two games which give rise to A-envy free allocations. The first game was the divide and choose game whose solutions were Pareto-efficient when restricted to the class of A-envy free allocations, but in general failed to satisfy efficiency. The second game was the equal division divide and choose game, whose solution always turned out to be Pareto-efficient. However, the divider seemed to be at an advantage in the first game than in the second. From the point of view of egalitarianism also EDDC fared worse than any other individually rational choice correspondence. Thus both games have something to recommend them although from the stand point of efficiency EDDC fares better.

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