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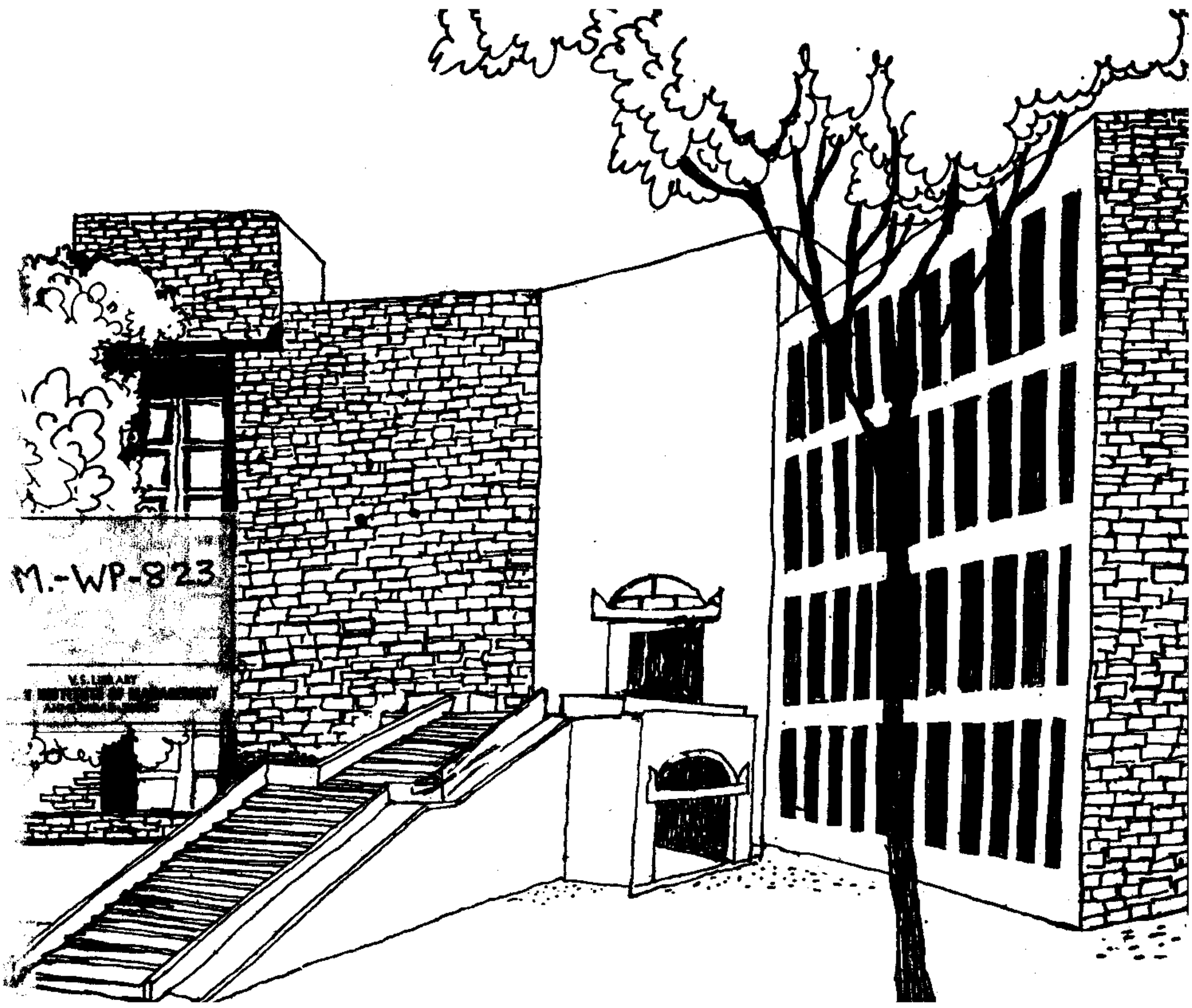

RATIONALIZATION OF BARGAINING SOLUTIONS BY SYMMETRIC METRICS AND RESPECT FOR UNANIMITY

By

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ABSTRACT

In this paper we show that rationalizability of bargaining solution by a symmetric metric implies that th bargaining solution is anonymous . We further show that rationa zability of a bargaining soluition by a metric implies that e solution satisifies metric respect for unanimity.

1. Introduction : We will follow the approach in defining an *n-person bargaining game* ($n \geq 2$) initiated by Nash (1950). Formally an *n-person bargaining game* S is a proper subset of R^n satisfying:

(1) S is closed, convex and $\sup \{x_i \ / \ x \in S\} \in R$ for all $i \in \{1,2,...,n\}$

(2) $0 = (0,0,...,0) \in S$ and $x > o$ for some $x \in S$

(3) S is comprehehnsive, i.e. for all $x \in S$ and $y \in R^n$, if $y \leqq x$, then $y \in S$

Let B donote the family of all bargaining games. When interpreting an $S \in B$, one must think of the following game situation. 'n' players (bargainers) may cooperate and agree on a feasible outcome x in S, giving utility x_i to player $i=1,...,n$, or they may fail to cooperate, in which case the game ends in the disagreement game outcome o. So for any $S \in B$, the disagreement outcome is fixed at o (which allows us to omit the usual axiom of translation invariance for bargaining solutions). Closedness of S is required for mathematical convenience; convexity stems from allowing lotteries in an underlying bargaining situation. Further it is assumed that S is bounded from above but not from below since we allow free disposal of utility. The requirement $x > o$ for some $x \in S$ serves to give each player an incentive to cooperate. Not all of the restrictions in (1) – (3) are necessary for all of our results, but assuming them simplifies matters.

Following Kaneko (1980) we define a bargaining solution as correspondence $\emptyset : B \to\to R^n$ assigning to each $S \in B$ a non emp subset $\emptyset(S) \subseteq S$ and such that Axiom o holds:

Axiom-o:– $\emptyset(S)$ depends only on (the shape of S)

Axiom o states explicitly that $\emptyset$ does not depend on an underlying bargaining situation (i.e. a set of lotteries and, a pair of utility functions mapping these into the plane). By most authors, this is implicitly assumed or taken for granted (however, cf Shapley (1969)}.

We also require our solution $\emptyset$ to satisfy the following condition:

(SIR): For each $S \in B$, for each $x \in \emptyset(S)$, $x_i > o \; \forall \; i \in \{1, \ldots, n\}$

(strict individual rationality)

In Lahiri (1989) we represent bargaining solutions by means of a metric which is defined on bargaining games, whereby the solutiuons are precisely those payoffs which are closest to being unanimously highest. In general, the purpose of a metric is to define distance and the metric generated by a bargaining solution defines the notion of a solution being close to awarding the highest payoff to all the players.

2. Metrizable Bargaining Solutions: Let $x = (x_1, \ldots, x_n) \in R^n$; let us agree to denote $\{y \in R^n / y_i \leq x_i, \forall i \in N\}$ by $S(x)$. Such games are called unanimity games and N obviously refers to the player set $\{1,..,n\}$

Definition 1: A bargaining solution $\emptyset$ is Paretian if $\forall\ x \in R^n_{++}$, $\emptyset(S(x))$:

Let $\bar{d}$ be a metric on B. Let $S_+ = S \setminus \{o\}$ and $S_{++} = S \cap R^n_{++}$

Definition 2: The metric $\bar{d}$ on B is a rationalization according to unanimity (henceforth a rationalization) for the bargaining solution $\emptyset$, if $\forall S \in B$, $\emptyset(S) = \{x \in S_{++} / \bar{d}(S, S(x)) \leq \bar{d}(S, S(y)) \forall y \in S_{++}\}$

That is, the metric $\bar{d}$ rationalizes $\emptyset$ according to the unanimity criterion whenever for any bargaining game S, the solution is the payoff whose convex comprehensive hull is the unanimity game ``nearest`` to the game and the payoff itself belongs to S. The characterization of the family of bargaining solutions having such a metric rationalization is provided by the following theorem:

Theorem 1: A bargaining solution $\emptyset$ has a metric rationalization if and only if it is Paretian.

Proof : {See Lahiri (1989)}.

A reasonable assumption for most bargaining solutions is that it satisfies weak Pareto optimality.

(WPO) for each $S \in B$, for each $x \in \emptyset(S)$, $y \in R^n_+$, $y_i > x_i$ for all

If some payoff vector awards the highest payoff to all the players then surely it should be declared the consensus solution. This is the unanimity principle which is naturally very appealing and which is satisfied by the Nash (1950) solution, the Kalai-Smorodinsky (1975) solution, the Yu (1973) solution, although not by the Kalai (1977) egalitarian solution. For 'most bargaining problems however, a unanimously preferred payoff vecctor generally does not exist, in as much as that 'most' bargaining problems are not representable as the comprehensive, convex hull of a single payoff vector. In Lahiri (1989) we consider the problem of finding out in what precise sense different solutions attempt (if at all) to approximate the ideal of using the unanimity rule.

An interesting property satisfied by many bargaining solutions is anonymity

(A) Let $S \in B$ and $\pi : \{1,\ldots,n\} \rightarrow \{1,\ldots n\}$ be any permutation.
For $x \in R^n$, $x = (x_1 \ldots, x_n)$ let $x_\pi \in R^n$ be defined as $X_\pi = (x_{\pi(1)}, \ldots, x_{\pi(n)})$ and $S_\pi = \{x_\pi \in R^n / x \in S\}$. Then $\emptyset(S_\pi) = \{\emptyset(S)\}_\pi$ (i.e. nothing is affected by renaming the players).

In this paper we find conditions that a metrizable bargaining solutions needs to satisfy in order to be anonymous.

for all $i \in N$ implies $y \notin S$

Let $W(S) = \{x \in S \ / \ y \in R^n_+, \quad y_i > x_i \ \forall \ i \in N \Rightarrow \ y \notin S\}$

A solution $\emptyset$: $B \to \to R^n_+$ satisfying (WPO) and (SIR) is called an <u>efficient bargaining solution</u>

For efficient bargaining solutions we have the following metric characterization:

<u>Theorem 2</u> : An efficient bargaining solutions $\emptyset$ has a metric rationalization if and only if it is Paretian.

<u>Proof</u> : {See Lahiri (1989)}.

The Nash (1950), Kalai-Smorodinsky (1975) and Yu (1973) solutions are Paretian and hence are metric rationalizable. The Kalai (1977) solution is not Paretian and hence not metric rationalizable.

3. <u>Symmetric Metric Rationalizations</u> : In this section we shall obtain metrizability conditions which lead to anonymous bargaining solutions.

Let $\bar{d}$ be a metric on B. We say that $\bar{d}$ is a symmetric metric if for all permutation π : $N \to N$ and for all games S, $T \in B$, $d(S,T) = d\ (S_\pi,\ T_\pi)$.

We now define the concept of <u>rationalization by a symmetric metric.</u>

<u>Definition 3:</u> A bargaining solution $\phi : B \twoheadrightarrow R^n$ is <u>rationalizable by a symmetric</u> metric if there exists a symmetric metric d on B, such that for any

$S \in B$, $\phi(S) = \{x \in S_{++} / d(S(x), S) \leq d(S(y), S)$ for any $y \in S_{++}\}$

i.e the symmetric metric on unanimity games rationalizes ϕ according the unanimity criterion whenever, for any game S, the solution set of S is given by the payoffs x such that the unanimity game S(x) is closest to S with respect to the given metric.

We also introduce the notion of <u>metric respect for unanimity.</u>

Definition 4 : A bargaining solution $\phi : B \twoheadrightarrow R^n$ has a <u>metric respect for unanimity</u> if there exists a metric $\bar{d}$ on B and a metric m on R^n such that for any game $S \in B$ and vectors $x, y \in S_{++}$, $d(S, S(x)) < d(S, S(y))$ implies that $m(\phi(S), x) \equiv \min_{z \in \phi(S)} m(z, x) \leq \min_{z \in \phi(S)} m(z, y) \equiv m(\phi(S), y)$

Notice that if ϕ has a <u>metric respect for unanimity</u>, then it is Paretian. The appeal of this property of proximity preservation lies in it offering a natural and consistent conception of a bargaining solution as a mechanism attempting to approximate or respect the social ideal of using the unanimity rule.

<u>Theorem 3</u>: If a bargaining solution $\phi : B \twoheadrightarrow R^n$ is rationalizable by a symmetric metric, then it satisfies the anonymity property, and it has a metric respect for unanimity.

Proof : Let ϕ be rationalizable by a symmetric metric. Then there exists a metric **d** on B such that for all $S,T \in B$, $d(S,T) = d(S_\pi, T_\pi)$ and for all $S \in B$

$\phi(S) = \{x \in S_{++} \,/\, d(S(x),S) \leq d(S(y),S) \;\forall\; y \in S_{++}\}$

Hence $(\phi(S))_\pi = \{x_\pi \in (S_{++})_\pi \,/\, d(S_\pi(x_\pi), S_\pi) \leq d(S_\pi(y_\pi), S_\pi) \;\forall\; y_\pi \in (S_{++})_\pi\}$

$= \{x \in (S_\pi)_{++} \,/\, d(S_\pi(x), S_\pi) \leq d(S_\pi(y), S_\pi) \;\forall\; y \in (S_\pi)_{++}\}$

$= \phi(S_\pi)$

Hence ϕ is anonymous.

To show that ϕ has a metric respect for unanimity, let us define the metric on $\mathbb{R}^n$ as follows: for any $x,y \in R^n$

$$m(x,y) = \begin{cases} 0 \text{ if } x = y \\ 1 \text{ if } x \neq y \end{cases}$$

Now consider a game S in B and two alternatives x, z in R^n such that $d(S,S(x)) < d(S,S(z))$. Here there are two possibilities:

(i) $x \in \phi(S)$, $z \notin \phi(S)$ in which case

$d(S,S(x)) < d(S,S(z))$, since d rationalizes ϕ

In turn

$m(\phi(S),x) = m(x,x) = 0 < m(\phi(S),z) = 1$

(ii) $x \notin \phi(S)$, $z \notin \phi(S)$

Here $m(\phi(S),x) = m(\phi(S),z) = 1$

In both cases, we obtain that $m(\phi(S),x) \leq m(\phi(S),z)$

For efficient bargaining solutions we have the following analogous characterization :

Definition 5: An efficient bargaining solution $\emptyset : B \rightarrow\rightarrow R^n$ is <u>rationalizable by a symmetric</u> metric if there exists a symmetric metric d on B, such that for any $S \in B$,

$$\emptyset(S) = \{x \in W(S) / d(S(x),S) \leq d(S(y),S) \ \forall y \in W(S)\}$$

Definition 6: An efficient bargaining solution $\emptyset : B \rightarrow\rightarrow R^n$ has a <u>metric respect for unanimity</u> if there exists a metric d on B and a metric m on R^n such that for any game $S \in B$ and vectors $x,y \in W(S)$, $d(S,S(x)) < d(S,S(y))$ implies that $m(\emptyset(S),x) \equiv \min_{z \in \emptyset(S)} m(z,x) \leq \min_{z \in \emptyset(S)} (z,y) \equiv m(\emptyset(S),y)$

<u>Theorem 4 :</u> If an efficient bargaining solution $\emptyset : B \rightarrow\rightarrow R^n$ is rationalizable by a symmetric metric, then it satisfies the anonymity property, and it has a metric respect for unanimity.

<u>Proof :</u> Analogous to the proof of Theorem 3.

<u>Conclusion :</u> In this paper we have shown that rationalizability by symmetric metric implies that the associated bargaining solution is anonymous. We know for instance that the non-symmetric Nash bargaining solutions as defined by Harsanyi and Selten (1972) or Kalai (1977 b) are not anonymous.For more on this see Peters (1988). Hence they are not rationalizable by a symmetric metric. However, the non-symmetric

Nash solution is Paretian and is therefore amenable to metric rationalizability. We also show that metric rationalization of bargaining solutions implies metric respect for unanimity.

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